Response of a Hexagonal Granular Packing under a Localized External Force: Exact Results

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Abstract.

We study the response of a two-dimensional hexagonal packing of massless, rigid, frictionless spherical grains due to a vertically downward point force on a single grain at the top layer. We use a statistical approach, where each mechanically stable configuration of contact forces is equally likely. We show that this problem is equivalent to a correlated q-model. We find that the response is double-peaked, where the two peaks, sharp and single-grain diameter wide, lie on the two downward lattice directions emanating from the point of the application of the external force. For systems of finite size, the magnitude of these peaks decreases towards the bottom of the packing, while progressively a broader, central maximum appears between the peaks. The response behaviour displays a remarkable scaling behaviour with system size N: while the response in the bulk of the packing scales as $\frac{1}{N}$, on the boundary it is independent of N, so that in the thermodynamic limit only the peaks on the lattice directions persist. This qualitative behaviour is extremely robust, as demonstrated by our simulation results with different boundary conditions. We have obtained exact expressions of the response and higher correlations for any system size in terms of integers corresponding to an underlying discrete structure.

Keywords: Granular statics, Response function

1. Introduction

Static properties of granular packings have been a very active field of research in recent years. Granular packings are assemblies of macroscopic particles that interact only via mechanical repulsion mediated through physical contacts [1, 2]. In contrast with continuum solids, forces on individual grains in a granular packing can be directly accessed both experimentally [3–5] and numerically [6–9]. These forces are found to be organized in highly heterogeneous networks that depend strongly on construction history of the packing [10]. Statistical studies, motivated to deal with such history dependence and heterogeneity of the forces on individual grains, have identified two main global characteristics of static granular packings. First, the distribution for the magnitudes of inter-grain forces is very broad, with an exponential decay for large force magnitudes and a plateau at small force magnitudes [3, 6]. Secondly, the average response of the packings to a single vertically downwards external force depends strongly on the underlying geometry: in ordered packings, the applied force is mainly transmitted along the principal lattice directions emanating from the point of application of the force, while in disordered packings there is a single vertical propagation direction [4, 5, 7].

Although a large number of different theoretical models have been proposed to study these two global characteristics of granular packings, each of these models successfully accounts for at most one of them. For example, the so-called q-model [11] is a scalar lattice model that describes the fluctuations in force transmission within a granular packing at the first approximation. While this model does produce realistic distributions for the magnitudes of inter-grain forces (see Ref. [12] for a more detailed discussion), it predicts a diffusive response to a single external force in conflict with experiments [13]. On the other hand, vectorial extensions of the q-model [13], compatible with a more general "stress-only" approach [14], were shown to lead to stochastic wave equations. For weak disorder, these equations predict a ray-like propagation of stresses in agreement with experiments, but for strong disorder, the corresponding behaviour of the stresses is less clear-cut. This has further led to the introduction of an ad-hoc "force chain splitting" model [15]. The vectorial extensions of the q-model and the force-chain splitting models do predict realistic response behaviour for granular packings, but the relation between microscopic stochasticity and resulting distributions for the magnitudes of inter-grain forces within these models is not entirely clear. Finally, in contrast with these stochastic approaches, classical elastic theory has been used for years in the engineering community [16, 17]. The predictions of this theory for the response behaviour of granular packings match the experimental results rather successfully [18]. However, elasticity theory provides only a macroscopic, average level description, and thus it provides no information on the force distribution. Moreover, its derivation from the grain-level mechanics still seems to be a significant challenge [16, 17].

From this perspective, a unifying approach leading to both realistic fluctuations in the individual inter-grain force magnitudes and transmissions of forces in a granular packing clearly seems to be necessary. Interestingly, in a different context (namely, that

of granular compaction) a basis for such an approach has been laid down by Edwards years ago [19, 20]. By analogy with conventional statistical mechanics, the fundamental hypothesis is to consider all "jammed" configurations equally likely. Although no microscopic justification for such an ergodic assumption is available so far, experiments and numerical studies of quasi-static granular media support such a thermodynamic picture [22]. It thus seems very tempting to extend this hypothesis to the study of forces in static granular packings by considering sets of forces belonging to all mechanically stable configurations equally likely.

If this idea of equal probability ensemble is to be applied to study the forces in a granular packing, it has to be realized that two levels of randomness are generally present in the ensemble of forces for stable granular packings [2]. This stems from the fact that forces in a granular packing depend critically on the underlying contact network between the grains. Contact networks that are deemed isostatic uniquely determine the forces admissible on them, and thus, the application of the equal-probability hypothesis to packings with isostatic contact networks amounts to considering each contact network equally likely. Such a consideration leads to wave equations in disordered geometry whach are in conflict with experiments [21]. However, realistic contact networks are generically non-isostatic, and the fact that several force configurations can be compatible with a given non-isostatic contact network gives rise to the second level of randomness [8, 9, 23].

Instead of applying Edward's hypothesis to both levels of randomness simultaneously, a natural first step is to apply the uniform probability hypothesis first to a fixed contact network, and then possibly average over various contact networks. Such a study in a fixed contact network has recently been shown to produce distributions for the magnitudes of inter-grain forces that compare very well with experiments [24]. In Ref. [25] we briefly showed that the application of the uniform probability hypothesis in an ordered geometry also leads to a response to an external force qualitatively in agreement with experiments. In this paper, we report the same study in full detail. More precisely, we determine the behaviour of the response of a two-dimensional hexagonal packing of rigid, frictionless and massless spherical grains placed between two vertical walls (see Fig.1), due to a vertically downward force $W_{\rm ext}$ applied on a single grain at the top layer. We define the response of the packing as $\left[\langle W_{i,j}\rangle - \langle W^{(0)}_{i,j}\rangle\right]/W_{\rm ext}$, where $W_{i,j}$ and $W^{(0)}_{i,j}$ are the vertical forces transmitted by the (i,j)-th grain to the layer below it respectively with and without applied external force $W_{\rm ext}$. For massless grains $W^{(0)}_{i,j} \equiv 0$. The angular brackets here denote averaging with equal probability over all configurations of mechanically stable non-negative contact forces.

We find that the problem of equally-probable force configurations in the hexagonal geometry is equivalent to a correlated q-model. Using this formulation, simulations for a variety of boundary conditions show that the response to an external force applied on the top of the packing displays two symmetric peaks lying precisely on the two downward lattice directions emanating from the point of application of the force. Moreover, the response exhibits remarkable scaling with the system size, implying that in the

thermodynamic limit the peaks propagate all the way to the bottom of the packing. Surprisingly, average values $\langle W_{i,j} \rangle$ as well as higher correlations can be calculated *exactly* for any system size via this formulation. We show that all these quantities can be expressed in terms of integers corresponding to an underlying discrete structure.

The paper is organized as follows. In Sec. 2, we examine the equal-probability hypothesis in a fixed geometry, and its application to the hexagonal geometry in detail. In Sec. 3, we numerically study the response as well as the distributions of the single q's and correlations between them. In Sec. 5 we detail the full theoretical calculation of $\langle W_{i,j} \rangle$ and higher correlations for any system size. We finally end this paper with a discussion in Sec. 6.

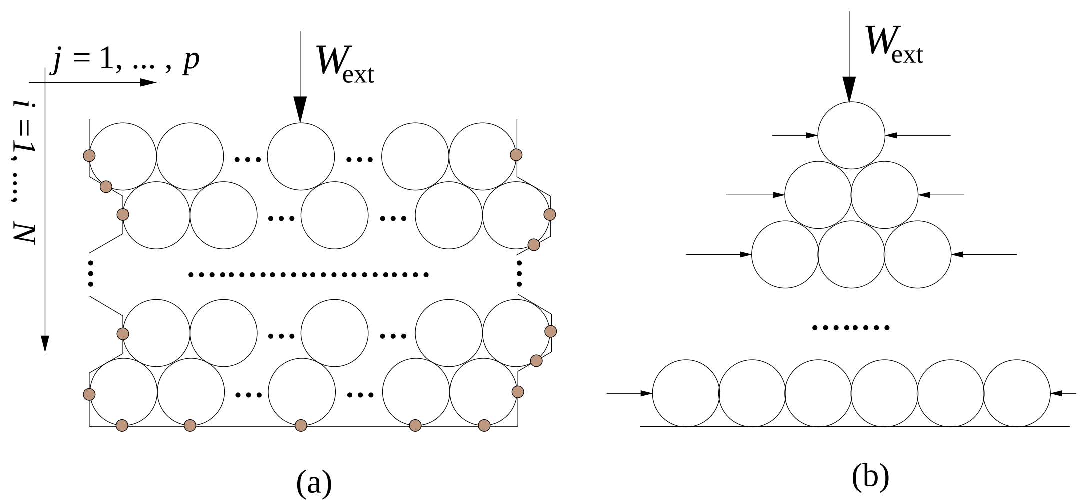

Figure 1. Our model: (a) $N \times p$ array of hexagonally close-packed rigid frictionless spherical grains in two-dimensions. At the top, there is only a single vertically downward point force applied on one grain. (b) For massless grains, the vertical forces are non-zero only inside a triangle formed by the two lattice directions emanating from the point of application of W_{ext}, so that we can restrict our study to that domain. The horizontal forces on boundaries of the triangular domain are the same as those on the boundaries in (a).

2. Uniform Measure on the Force Ensemble

2.1. Force Ensemble in Generic Packings

To start with, let us examine more closely the inter-grain forces in a granular packing [27]. As pointed out earlier, the grains mutually interact only via (mechanical) repulsive contact forces. The contacts are assumed pointlike, and in the present study, we consider only frictionless grains, so that each contact force is locally normal to the grain surface. Thus, the most fundamental entity entering the description of forces in a granular packing is the *contact network*, i.e., the set of all contact points with directions normal to the grain surfaces at respective contacts [20].

Consider such a contact network formed by a packing of P grains with Q contacts. For simplicity, we restrict ourselves to two dimensions in the following analysis. At each contact k for $1 \leq k \leq Q$, the force is represented by its magnitude F_k along the locally normal direction to the grain surface, with the convention that a positive magnitude of F_k corresponds to a repulsive force. The forces applied on a given grain must satisfy three Newton's equations: two for balancing forces in the x and y directions and one for balancing torque. For the system in its entirety, the contact forces can be grouped in a column vector F consisting of Q non-negative scalars $\{F_k\}$, satisfying $S = 3P$ stability equations represented by $\mathbf{A} \cdot \mathsf{F} = \mathsf{F}_{\text{ext}}$. Here, F_{ext} is an S-dimensional column vector representing the external forces and torques on the grains, and $\mathbf{A}$ is an $S \times Q$ matrix uniquely specified by the contact network. If the grains in question are disks, as is the case in most of theoretical and numerical studies, the torque balance is automatically satisfied, and the total number of equations S drops to $2P$. Note also that there is some freedom in the definition of F: a contact force between a grain and a boundary can either be considered as an internal force, i.e., as a part of F; or as an external force, in which case it becomes a part of F_{ext}.

In the above description, if $\mathbf{A}$ is invertible — which of course implies $Q = S$ — the force configuration allowed on the contact network is unique: such a contact network is called isostatic. Otherwise either there is an extended set $\mathcal{E}$ of allowed force configurations, in which case the network is called hyperstatic, or there is none, the packing under consideration is unstable under the imposed external forces.

Theoretical arguments suggest that perfectly rigid grains generically form isostatic packings [26]. Physical grains are of course never infinitely rigid, and the corresponding contact networks are hyperstatic. These observations are confirmed by numerical studies, which moreover find that the convergence to isostaticity with increasing rigidity of the grains is rather slow [23]. To determine the forces uniquely, one then in principle has to take into account the "force law", which relates the deformation of a particle to the force applied on it, as well as the construction history of the packing.

Nevertheless, some macroscopic properties of a granular packing, such as the ditribution of force magnitudes and the shape of the average response to a point force, are independent of the details of the force law [24]. From that point of view, if one considers a packing of grains of large but finite rigidity, the deformations of the grains are small with respect to their characteristic sizes, and these deformations could be altered without essentially modifying the (hyperstatic) contact network [24]. Experimentally, this can be achieved by gently tapping the packing without adding or removing contacts [2]. Without having to delve deeper in the microscopic force laws, one can then analyze the contact force configurations as a subset of the set $\mathcal{E}$ of allowed force configurations. By analogy to classical statistical mechanics, one can study the statistical properties of the set $\mathcal{E}$. A natural starting point is to assume that any point in $\mathcal{E}$ is visited with the same frequency, similar to a microcanonical ensemble. In other words, one assigns a uniform probability measure on $\mathcal{E}$, under which all allowed contact force configurations are equally likely. We should however keep in mind that *a priori* there is no clear

justification for such an ergodic hypothesis.

The uniform measure on $\mathcal{E}$ can be defined more precisely. The set of all solutions of $\mathbf{A} \cdot \mathsf{F} = \mathsf{F}_{\text{ext}}$ is an affine space, whose dimension is the dimension of the kernel of $\mathbf{A}$, and $\mathcal{E}$ is a subset of that space where $F_k \geq 0 \; \forall k = 1 \ldots Q$. It can be shown that $\mathcal{E}$ is usually a compact polygon [8], so that the uniform measure is well defined. For a hyperstatic packing, $Ker(A)$ is non-zero, and a parametrization of $\mathcal{E}$ can be constructed via the three following steps: (1) one first identifies an orthonormal basis $\{\mathsf{F}^{(l)}\}$ $(l = 1, \ldots, d_K = Q - S)$ that spans the space of $Ker(\mathbf{A})$; (2) one then determines a unique solution $\mathsf{F}^{(0)}$ of $\mathbf{A} \cdot \mathsf{F}^{(0)} = \mathsf{F}_{\text{ext}}$ by requiring $\mathsf{F}^{(0)}.\mathsf{F}^{(l)} = 0$ for $l = 1, \ldots, d_K$; and (3) one finally obtains all solutions of $\mathbf{A} \cdot \mathsf{F} = \mathsf{F}_{\text{ext}}$ as $\mathsf{F} = \mathsf{F}^{(0)} + \sum_{l=1}^{Q-S} f_l \, \mathsf{F}^{(l)}$, where f_l are real numbers. The restriction of the f_l's to a set $\mathcal{S}$ generating non-negative forces is a possible parametrization of $\mathcal{E}$. The uniform measure on $\mathcal{E}$ is thus equivalent to the uniform measure $d\mu = \prod_k dF_k \, \delta(\mathbf{A} \cdot \mathsf{F} - \mathsf{F}_{\text{ext}}) \, \Theta(F_k) = \prod_l df_l$ on $\mathcal{S}$.

2.2. Force Ensemble in the Hexagonal Geometry

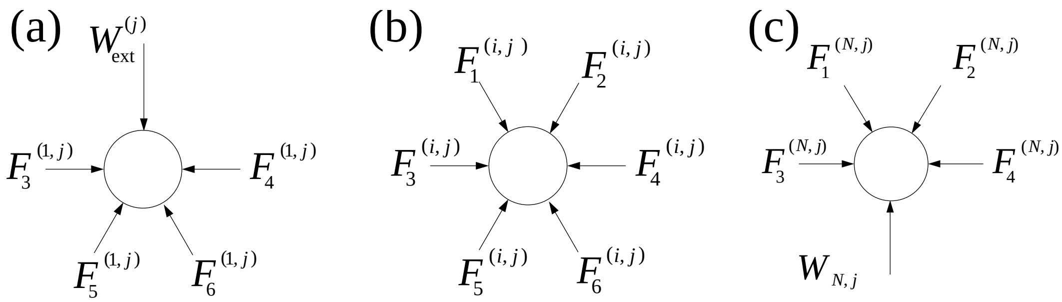

Figure 2. Schematically shown forces on the j-th grain in the i-th layer: (a) $i = 1$, (b) $i \leq N$ and (c) $i = N$; $F_m^{(i,j)} \geq 0 \; \forall m$.

We will now apply the general method described in Sec. 2.1 to a hexagonal packing of monodisperse, rigid and frictionless disks under the localized external force W_{ext} (see. Fig. 1). We will concentrate on massless particles since considering masses does not fundamentally change the qualitative behaviour [25]. Our aim is to calculate the mean of $W_{i,j} = \frac{\sqrt{3}}{2}\left[F_1^{(i,j)} + F_2^{(i,j)}\right]$ (see Fig. 2) by considering all admissible force configurations to be equally likely.

2.2.1. Force balance on individual grains: To start with, we calculate the forces $F_5^{(i,j)}$ and $F_6^{(i,j)}$ for each grain in terms of other forces by using Newton's equations. For the top layer, i.e., $i = 1$ [Fig. 2(a)],

$$
\begin{aligned}
F_5^{(1,j)} &= \frac{1}{\sqrt{3}} W_{\text{ext}}^j + [F_4^{(1,j)} - F_3^{(1,j)}] \\
F_6^{(1,j)} &= \frac{1}{\sqrt{3}} W_{\text{ext}}^j - [F_4^{(1,j)} - F_3^{(1,j)}] \,,
\end{aligned} \tag{1}
$$

while for a grain in the bulk [Fig. 2(b)]

$$\begin{aligned} F_5^{(i,j)} &= F_2^{(i,j)} + [F_4^{(i,j)} - F_3^{(i,j)}] \\ F_6^{(i,j)} &= F_1^{(i,j)} - [F_4^{(i,j)} - F_3^{(i,j)}]\,, \end{aligned} \tag{2}$$

and finally for the bottom layer [Fig. 2(c)],

$$\begin{aligned} W_{N,j} &= \frac{\sqrt{3}}{2}\left[F_1^{(N,j)} + F_2^{(N,j)}\right] \\ F_4^{(N,j)} - F_3^{(N,j)} &= \frac{1}{2}\left[F_1^{(N,j)} - F_2^{(N,j)}\right]\,. \end{aligned} \tag{3}$$

In our study, a vertically downward force W_{ext} is applied on a single grain j_0 in the top layer, i.e., $W_{(1,j)} = W_{\text{ext}}\delta_{j,j_0}$. Equations (1-3) then show that this force can propagate only inside a triangle formed by the two downward lattice directions emanating from the grain j_0. Outside the triangle, all non-horizontal forces are zero, and for the horizontal ones, $F_3^{(i,j)} = F_4^{(i,j)}$. Since our main interest are the $W_{(1,j)}$'s, this implies that we can restrict ourselves to the triangular domain, for which the forces exerted on the boundaries are the same as the forces on the boundary of the original system [see Fig. 1(b)].

2.2.2. Parametrization of the force ensemble: As stated in Sec. 2.1, F contains one scalar (force magnitude) for each inter-grain contact. This is a consequence of the action-reaction principle, which gives the following identifications for $1 < i \leq N$ and $1 < j \leq i$:

$$F_1^{(i,j)} = F_6^{(i-1,j-1)}, \qquad F_2^{(i,j)} = F_5^{(i-1,j)}, \qquad \text{and} \qquad F_3^{(i,j)} = F_4^{(i,j-1)}\,. \tag{4}$$

We consider two different vertical boundaries: (i) "hard walls" described in Fig. 1 and (ii) periodic boundaries in the j direction. While in both cases the contact network is clearly hyperstatic, the configurations of the internal forces are slightly different. We start with the case of "hard walls".

For reasons which will become clear later on, we will consider the contact forces on the right and the bottom boundaries as internal forces, i.e., as a part of F, while the forces on the top and right boundaries will be considered as a part of F_{ext}. Simple counting then shows that $Q = \frac{3}{2}N^2 + \frac{1}{2}N$, while the number of equations is $N(N+1)$, implying that $d_K = \frac{N(N-1)}{2}$.

Notice from Eqs. (1-2) that in this description, the lateral forces $F_3^{(i,j)}$ and $F_4^{(i,j)}$ enter the equations for the non-horizontal forces only via their difference $G_{i,j} = F_4^{(i,j)} - F_3^{(i,j)}$ for $i = 1 \ldots N-1$ and $j = 1 \ldots p$. A natural parametrization of $\mathcal{E}$ is given by these $G_{i,j}$'s: once the $G_{i,j}$ are fixed, all the other non-horizontal forces can be uniquely determined by solving Eqs. (1-3) layer by layer from top down. It is easily seen that the number of these parameters is indeed d_K, and that they correspond to an orthonormal basis of $Ker\,\mathbf{A}$.

Thus, the set $\mathcal{E}$ is parametrized by the $G_{i,j}$'s, and the $G_{i,j}$'s are restricted within a subset $\mathcal{S}$ in order to keep all internal forces non-negative. The non-negativity of $F_5^{(i,j)}$

and $F_6^{(i,j)}$ implies

$$-\frac{1}{\sqrt{3}}W_{\text{ext}} \leq G_{1,j_0} \leq \frac{1}{\sqrt{3}}W_{\text{ext}} \qquad \text{and} \qquad -F_2^{(i,j)} \leq G_{i,j} \leq F_1^{(i,j)}\,. \tag{5}$$

Furthermore, the horizontal forces can be expressed as

$$F_4^{(i,j)} = F_3^{(i,1)} + \sum_{j'=1}^{j} G_{i,j'}\,, \tag{6}$$

where $F_3^{(i,1)}$'s are *external* forces. If the magnitudes of $F_3^{(i,1)}$'s are taken to be larger then $\frac{2}{\sqrt{3}}W_{\text{ext}}$, then it is easy to see that $F_4^{(i,j)}$'s remain positive for *all* values of $G_{i,j}$'s that satisfy inequality (5); otherwise, to have $F_3^{(i,j)} \geq 0$, the available range for each $G_{i,j}$ would have to be further restricted within the bounds defined by the inequality (5). In this paper, we concentrate on the case $F_3^{(i,1)} = \frac{2}{\sqrt{3}}W_{\text{ext}}$ where the full range (5) is allowed. We will consider the case $F_3^{(i,1)} < \frac{2}{\sqrt{3}}W_{\text{ext}}$ briefly in Sec. 3.1.

Notice that the positivity conditions for $F_4^{(i,j)}$'s defined via Eq. (6) is the only place where the precise values of $F_3^{(i,1)}$'s enter the analysis. Clearly, considering $F_3^{(i,1)}$'s as a part of F would lead to them being parameters of $\mathcal{E}$ (and thus an unbounded $\mathcal{S}$ due to the lack of upper bounds of $F_3^{(i,1)}$'s). We therefore choose $F_3^{(i,1)}$'s as external forces with *fixed* values $\frac{2}{\sqrt{3}}W_{\text{ext}}$ to keep $\mathcal{S}$ bounded and the uniform measure well-defined.

With the above convention, the set $\mathcal{S}$ is an $\frac{(N-1)N}{2}$-dimensional polyhedron in the $G_{i,j}$-space defined by inequalities (5). The response of the packing to the external force W_{ext} is then defined as

$$\langle W_{i,j} \rangle \equiv \frac{1}{\mathcal{N}} \int_{\mathcal{S}} W_{i,j} \prod_{k,l} dG_{k,l}; \tag{7}$$

i.e., the averages $\langle \ldots \rangle$ are defined over an ensemble of force configurations, where each force configuration is equally likely, with $\mathcal{N} = \int_{\mathcal{S}} \prod_{k,l} dG_{k,l}$ the normalization constant.

Most of the previous analysis remains valid for periodic boundaries as well. The main difference is that the $F_3^{(i,1)}$'s are now internal forces (i.e., a part of F), as $F_3^{(i,1)} = F_4^{(i,p)}$ due to the action-reaction principle. The phase space $\mathcal{E}$ can again be parametrized by $G_{i,j}$ for $i \leq N, 1 \leq j \leq N$, with the additional constraint that

$$\sum_{j=1}^{p} G_{i,j} = 0 \qquad \forall i = 1 \ldots N. \tag{8}$$

Once again, only the $G_{i,j}$'s physically enter the problem, and the precise values of the horizontal forces are immaterial. The horizontal forces are well-defined only if one *chooses* fixed reference values for, say, the $F_3^{(i,1)}$'s. If these values are large enough, as explained in Eq. (6) and therebelow, all the horizontal forces are positive irrespective of the $G_{i,j}$ values within the bounds defined by inequality (5). Then the set $\mathcal{S}_p$ of allowed values of the parameters $G_{i,j}$ is the intersection of the hyperplane $\mathcal{S}_h$ defined by Eq. (8) and the polyhedron defined by the inequality (5).

2.2.3. The q-coordinates: There is an alternative formulation of this problem in terms of new variables

$$q_{i,j} = \frac{\sqrt{3}(G_{i,j} + F_2^{(i,j)})}{2W_{i,j}}\,, \tag{9}$$

where $q_{i,j}$ is the fraction of $W_{i,j}$ that the (i,j)th particle transmits to the layer below it towards the left, i.e., $F_5^{(i,j)} = \frac{2}{\sqrt{3}} q_{i,j} W_{i,j}$ and $F_6^{(i,j)} = \frac{2}{\sqrt{3}}(1 - q_{i,j})W_{i,j}$. The mapping (9) then reduces inequality (5) to $0 \le q_{i,j} \le 1$. Clearly, $W_{1,j_0} = W_{\rm ext}$ is the external force applied on the top layer. For $i > 1$, the $W_{i,j}$'s in the successive layers are related via

$$W_{i,j} = (1 - q_{i-1,j-1})\, W_{i-1,j-1} + q_{i-1,j}\, W_{i-1,j} \tag{10}$$

i.e., $W_{i,j}$ is a function of $q_{k,l}$ for $k < i$.

From the formulation in terms of the q's it may seem from Eq. (10) that in the hexagonal geometry of Fig. 1, one simply recovers the q-model [11]. There is however an important subtlety to take notice of. In the q-model, the q's corresponding to different grains are usually uncorrelated. In our case, the uniform measure on $\mathcal{E}$ implies, from Eq. (9), that

$$\prod_{i,j} dG_{i,j} = \left[\frac{2}{\sqrt{3}}\right]^{\frac{N(N-1)}{2}} \prod_{i,j} dq_{i,j}\, W_{i,j}(q)\,, \tag{11}$$

i.e., due to the presence of the Jacobian on the right hand side of Eq. (11), the uniform measure on $\mathcal{S}$ translates to a non-uniform measure on the $\frac{N(N+1)}{2}$-dimensional unit cube formed by the accessible values of the q's. In the case of periodic boundary conditions, the constraint (8) reduces to

$$\sum_{j=1}^{p} W_{i,j} q_{i,j} = \sum_{j=1}^{i-1} W_{i-1,j} q_{i-1,j} = \frac{1}{\sqrt{3}} W_{\rm ext}. \tag{12}$$

3. Numerical results

3.1. Shape and scaling of the response

We now present the results for the $\langle W_{i,j}\rangle$'s evaluated numerically by implementing detailed balance with respect to the probability measure $d\mu = \frac{1}{\mathcal{N}} \prod_{i,j} dq_{i,j}\, W_{i,j}(q)$ on the $\frac{N(N-1)}{2}$-dimensional unit cube in the q-space. At each step, a single q is modified, and the change is accepted with a Metropolis acceptance ratio. From Eq. (10), it is easy to see that the $\langle W_{i,j}\rangle$ values scale linearly with that of $W_{\rm ext}$. Thus, from now on, we set $W_{\rm ext} = 1$.

Our simulation results for $\langle W_{i,j}\rangle$ in the case of hard walls with $F_3^{(i,1)} = \frac{2}{\sqrt{3}} W_{\rm ext}$ for $N = 37$ are plotted in Fig. 3(a), using the built-in cubic interpolation function of Mathematica. Outside the triangle shown in Fig. 1(b), $\langle W_{i,j}\rangle \equiv 0$ appears in deep indigo; the largest $\langle W_{i,j}\rangle$ value within the triangle appears in dark red; and any other non-zero $\langle W_{i,j}\rangle$ value is represented by a linear wavelength scale in between. The

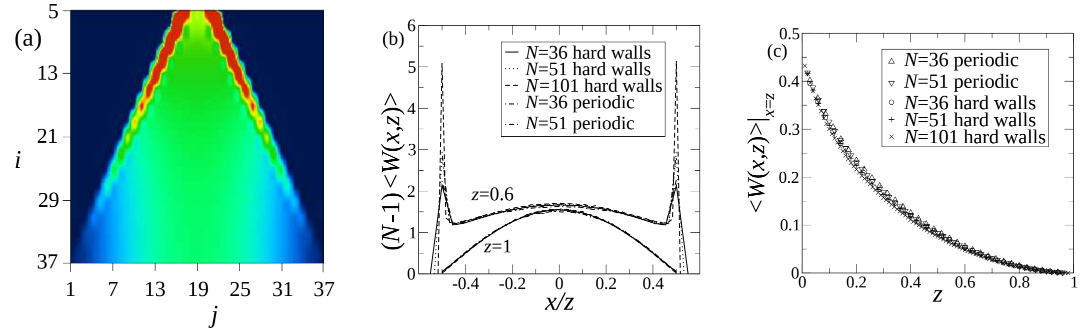

Figure 3. Simulation results for $\langle W_{i,j}\rangle$: (a) colour plot for $\langle W_{i,j}\rangle$ with hard walls; $N = 37$. The null vanishing forces appear in deep indigo, while the largest are represented in dark red. The other values are represented by a linear wavelength scale in between.(b)-(c) Behavior of $\langle W_{i,j}\rangle$ in reduced co-ordinates x and z for different N-values and boundary conditions: (b) scaling of $\langle W(x,z)\rangle$ with system size for $|x| < z$ at two z values; (c) data collapse for $\langle W(x,z)\rangle|_{x=z}$ for different N values and boundary conditions. See text for further details.

thin green regions that appear on the outer edge of the triangle are artifacts of the interpolation.

We find for any system size N that the $\langle W_{i,j}\rangle$ values display two symmetric peaks that lie precisely on the two downward lattice directions emanating from the point of application of the force W_{ext}. The width of these peaks is a single grain diameter and with depth their magnitudes decrease, while a broader central peak appears. At the very bottom layer ($i = N$) only the central maximum for $\langle W_{i,j}\rangle$ remains.

We further define $x = \frac{j-j_0}{N}$ and $\frac{j-j_0+1/2}{N}$ respectively for even and odd i, and $z = \frac{i}{N}$ [see Fig. 1] in order to put the vertices of the triangle formed by the locations of non-zero $\langle W_{i,j}\rangle$ values on $(0,0), (-1/2,1)$ and $(1/2,1)$ $\forall N$. The excellent data collapse indicates that the $\langle W_{i,j}\rangle$ values for $|x| < z$ scale with the inverse system size [Fig. 3(b); we however show only two z values], while the $\langle W_{i,j}\rangle$ values for $|x| = z$ lie on the same curve for all system sizes [Fig. 3(c)]. The data suggest that in the thermodynamic limit $N \to \infty$, the *response field* $\langle W(x,z)\rangle$ scales $\sim 1/N$ for $|x| < z$, but reaches a *non-zero* limiting value on $|x| = z$ $\forall z < 1$. We thus expect $\lim_{N\to\infty} \langle W(x,z)\rangle|_{|x|=z} > \langle W(x,z)\rangle|_{|x|<z}$ $\forall z < 1$; or equivalently, a *double-peaked response field at all depths $z < 1$ in the thermodynamic limit*.

The introduction of the additional constraint (12) that correspond to periodic boundaries modifies neither the qualitative, nor the scaling properties of the response function as can be observed in Fig. 3(b)-(c).

Although at all places in this paper we consider each side force $F_3^{(i,1)} = \frac{2}{\sqrt{3}}W_{\text{ext}}$ so that the full range of q-values are allowed for each $q_{i,j}$, in this paragraph, we take a short digression to discuss what happens when one chooses $F_3^{(i,1)}$'s to be smaller than $\frac{2}{\sqrt{3}}W_{ext}$. At the extreme limit when $F_3^{(i,1)} = 0$, it is easy to see that the only grains that correspond to non-zero $W_{i,j}$-values lie exactly on the boundary. From there on, it is intuitively clear that once the values of the $F_3^{(i,1)}$'s are pushed higher, the *finite* range

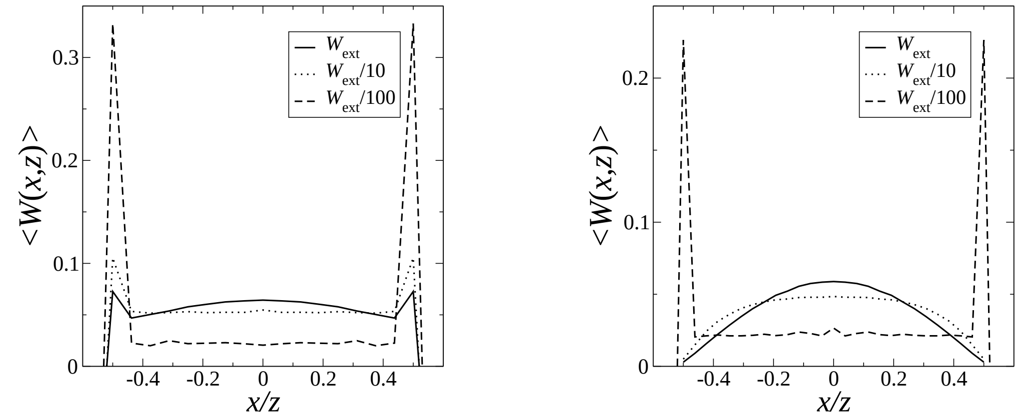

Figure 4. Responses in reduced coordinates for systems with three different values of $\frac{\sqrt{3}}{2}F_3^{i,1}$ at two different depths, with periodic boundary conditions: (a) at $z = 0.75$, (b) at $z = 1$.

of allowed q-values would begin to transfer some of the vertical forces into the bulk. A higher range of allowed values of the q's, caused by higher values of $F_3^{(i,1)}$'s, would thus effectively result in smaller peaks on the boundary and correspondingly, bigger fractions of the total vertical force within the bulk.

Fig. 4, where we plot the response at two different z values ($z = 0.75$ and $z = 1$) for $N = 51$ and $\frac{\sqrt{3}}{2}F_3^{(i,1)} = W_{\text{ext}}$, $W_{\text{ext}}/10$ and $W_{\text{ext}}/100$ respectively with periodic boundary conditions, clearly supports such an intuitive picture.

3.2. Linearity of response

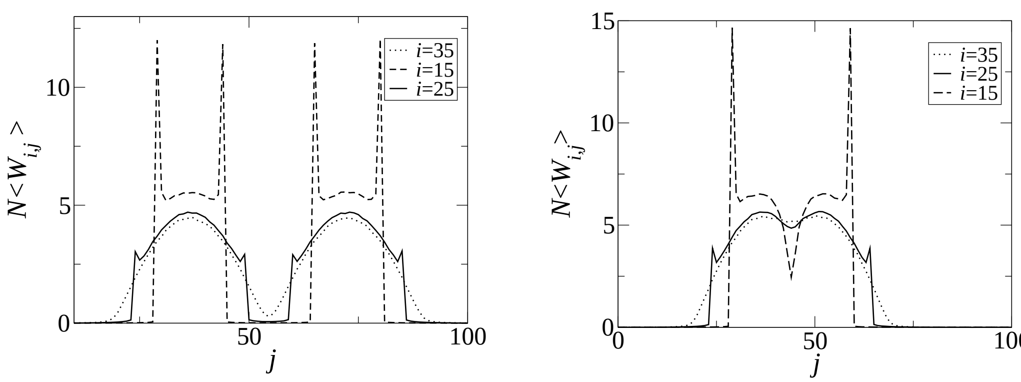

Figure 5. Simulation results for the response $N\langle W_{i,j}\rangle$ due to two vertically downward forces of unit magnitude each applied on the grains $(1, j_1)$ and $(1, j_2)$ for $N = 35$ and $p = 100$: (a) $j_2 - j_1 = 35$ (b) $j_2 - j_1 = 15$. Three different values of i are displayed in each case.

Since the values of the $\langle W_{i,j}\rangle$'s trivially scale linearly with the magnitude of W_{ext}, a natural question is whether the response depends linearly on F_{ext} in general, i.e., whether the response to a superposition of external forces is simply the superposition

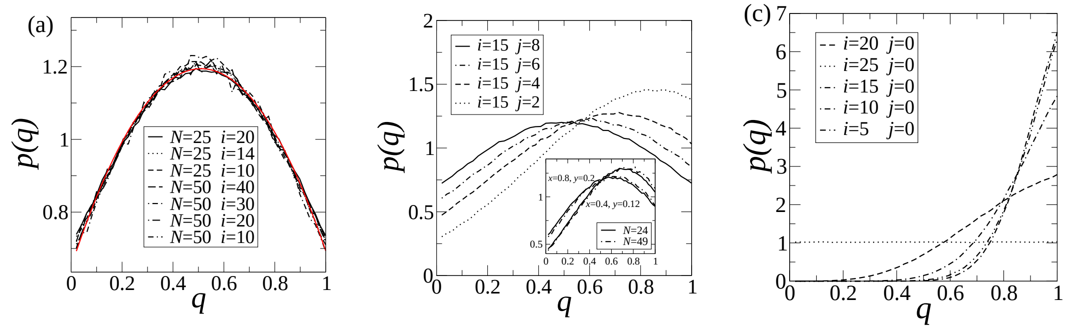

Figure 6. $p(q_{i,j})$ for (a) $j = i/2$ for $N = 25$ and 50, with a quadratic fit in red (b) for $N = 25$ at $i = 15$ for four different j values [inset: self-similarity of $p(q_{i,j})$ for $N = 25$ and $N = 50$ at $(x, z) = (0.12, 0.4)$ and$(x, z) = (0.2, 0.8)$], (c) for $N = 25$ on the left boundary $j = 0$ at different heights i.

of responses.

It might appear at first sight that the response is indeed linear — after all, the heart of the problem is the system of linear equations $\mathbf{A} \cdot \mathsf{F} = \mathsf{F}_{\rm ext}$. In fact, with the notations of Sec. 2.1, one might argue that since $\mathsf{F}^{(0)}$ is a linear function of $\mathsf{F}_{\rm ext}$, while $KerA$ is fixed for a given contact geometry, the averages over the affine space $\mathsf{F}^{(0)} + KerA$ should depend linearly on $\mathsf{F}_{\rm ext}$. Notice however that the set $\mathcal{E}$ corresponds *only* to $F_k \geq 0$, and the shape and volume of $\mathcal{E}$ in general depend on $\mathsf{F}_{\rm ext}$ in a complex way, and thus the response needs not be linear.

To illustrate this point, we apply two vertically downward forces of unit magnitude on the $(1, j_1)$th and $(1, j_2)$th grain for $N = 35$ and $p = 100$. The $|j_2 - j_1| > N$ case is a trivial situation, since in this case the two triangular regions of non-zero $W_{i,j}$'s do not overlap with each other, and the system is simply the superposition of two stochastically independent subsystems. In other words, the response observed for the $|j_2 - j_1| > N$ is simply a superposition of the responses of two individual responses [see Fig. 5(a)]. For $|j_2 - j_1| \leq N$ however, the two triangular regions do overlap and the subsystems interact; as shown in Fig. 5(b), the fact that there exists a central *minimum* for the response at $i = 15$ is a clear demonstration that the linearity of the responses does not hold.

3.3. q-distributions and correlations

We have just seen that the qualitative behaviour of our model is very different from that of the q-model. It thus seems reasonable that we study the difference between these two models at the level of individual grains.

Our starting hypothesis of equal probability for all mechanically stable force configurations implies that the joint probability distribution for the $q_{i,j}$'s is given by $\frac{1}{\mathcal{N}} \prod_{ij} W_{i,j}(q)$, where $\mathcal{N}$ is the normalization constant. This distribution does not factorize into a product of terms that depend on single $q_{i,j}$'s, and as a result, the $q_{i,j}$'s associated with different grains are correlated with each other. Below we numerically

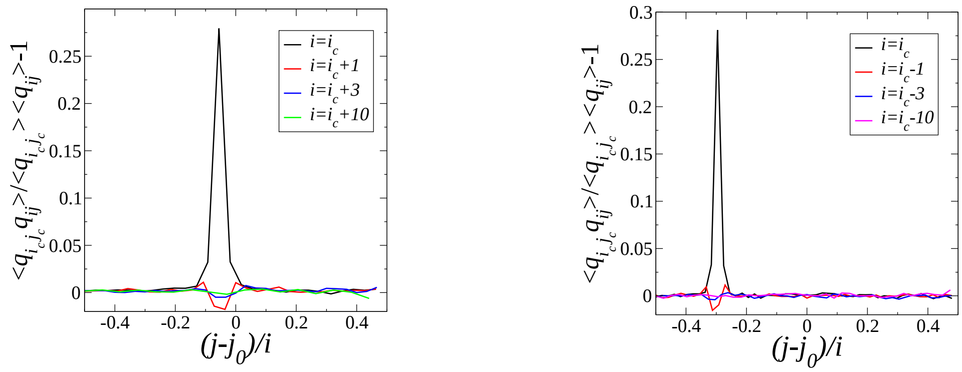

Figure 7. Correlations between q_{i_c,j_c} and $q_{i,j}$ at various values of i for $N = 50$, (a) $(i_c, j_c) = (25, 12)$ and (b) $(i_c, j_c) = (47, 10)$.

investigate the properties of the induced probability distributions $p(q_{i,j})$ for single $q_{i,j}$'s, as well as the correlations between different $q_{i,j}$'s throughout the system.

Since individual $W_{i,j}(q)$'s are independent of $q_{N,k}$'s for $k = 1, \ldots, N$, each of the $q_{N,k}$'s is uncorrelated with all other other q's in the system, and is also uniformly distributed within the interval $[0, 1]$. However, higher up in the packing, the q-distributions start to show a single maximum. For an odd i and $j_0 = \frac{i+1}{2}$, i.e., at the centre of the layer, $p(q_{i,j_0})$ exhibits a single maximum precisely at $q_{i,j_0} = 0.5$ due to symmetry reasons. Furthermore, $p(q_{i,j_0})$ is independent of the precise value of $i < N$ as well as the system size, and it is well-fitted by a quadratic polynomial [see Fig. 6(a)]. For a given layer i, the larger $|j - j_0|$ is, the more the location of the single maximum of $p(q_{i,j})$ shifts away (symmetrically) from $q_{i,j} = 0.5$: for $j - j_0 > 0$, this maximum occurs at $q_{i,j} > 0.5$ and for $j - j_0 < 0$, this maximum occurs at $q_{i,j} < 0.5$ [Fig. 6(b)]. Finally, exactly on the boundary the maximum of $p(q_{i,j})$ occurs at $q = 0$ (left boundary) or at $q = 1$ (right boundary) [Fig. 6(c)].

Interestingly, the self-similarity that we observed for the response behaviour also seems to be valid for $p(q_{i,j})$ in the bulk [see the inset of Fig. 6(b)]. This stands at a stark contrast to the physical behaviour of the q-model, as the self-similarity implies that in our model, the forces in a granular packing do *not* propagate from top down, instead they depend on the whole extent of the system.

In Fig. 7, we show results for the correlations between q_{i_c,j_c} and $q_{i,j}$ for several values of i and j for $N = 50$, $(i_c, j_c) = (25, 12)$ and $(i_c, j_c) = (47, 10)$. It appears from the plots that the correlations between the q's at different locations are very weak.

4. Exact Results for $\langle W_{i,j} \rangle$: Summary

Having presented the simulation results above, from this section onwards we concentrate on the exact (theoretical) results. It turns out that in this model not only the $\langle W_{i,j} \rangle$ values, but also all the higher moments and correlations of the $W_{i,j}$'s can be expressed exactly in terms of integers defined by simple recursion relations. The full calculational

details for the exact results for $\langle W_{i,j}\rangle$ will be presented in Sec. 5. The details of the procedure are slightly involved, and therefore we provide a layout summary of results and methods in this section.

Our main result are exact expressions for all moments of $W_{i,j}$ for any system size N in terms of integers defined by recursive relations. For instance, the zero-th moment or normalization constant $\mathcal{N}$ for a packing of N layers reads

$$\mathcal{N} = \frac{1}{\left[\frac{(N-1)N}{2}\right]!} \sum_{j_1,\ldots,j_{N-2}} \Lambda^{(N-2)}_{j_1,\ldots,j_{N-2}} \,. \tag{13}$$

where $\Lambda^{(N-2)}_{j_1,\ldots,j_{N-2}}$ are integers indexed by $\{j_k\}_{1\le k\le N-2}$, with $1 \le j_1 \le 2$ and $1 \le j_k \le k + j_{k-1}$. They are given by the following reccurrence relations

$$\begin{aligned} \Lambda^{(2)}_{j_1 j_2} &= \delta_{j_2,1} \\ \Lambda^{(p+1)}_{j_1,\ldots,j_{p+1}} &= \delta_{j_{p+1},1} \sum_{k_1,\ldots,k_p} \Lambda^{(p)}_{k_1,\ldots,k_p} \prod_{l=1}^{p} \Theta(j_{l+1} - k_l) \,. \end{aligned} \tag{14}$$

with Θ the discrete Heaviside (step) function.

Any higher moment of $W_{i,j}$ can be expressed as $\frac{1}{\mathcal{N}} \sum_{j_1,\ldots,j_{N-2}} \bar{\Lambda}^{(N-2)}_{j_1,\ldots,j_{N-2}}$, where $\bar{\Lambda}^{(N-2)}_{j_1,\ldots,j_{N-2}}$ are integers obtained through slight modifications of the reccurence formulas (14), described in full detail in sections 5.3.2, 5.3.3, and 5.4. These results establish a non-trivial equivalence between the model studied here and a discrete combinatorial problem defined in Sec.5.6.

The crux of calculating any moment of W_{ij} lies in the transformation of the integration measure $\prod_{i,j} dG_{i,j}$ [such as in Eq. (7)]. In Eq. (11), $\prod_{i,j} dG_{i,j}$ has been rewritten in terms of the $q_{i,j}$'s. With another change of variables from $q_{i,j}$'s to $W_{i,j}$'s, the integration measure can be further expressed as

$$\prod_{i,j} dG_{i,j} = \left[\frac{2}{\sqrt{3}}\right]^{\frac{N(N-1)}{2}} \prod_{i,j} dq_{i,j}\, W_{i,j}(q) = \left[\frac{2}{\sqrt{3}}\right]^{\frac{N(N-1)}{2}} \prod_{i,j} dW_{i,j} \,. \tag{15}$$

While such a variable change simplifies the integrands, the mapped volume $\mathcal{S}$ (see the last paragraph of Sec. 2.1) is however not an N-dimensional square anymore, but again a polygone. Despite this complication, one can build up a *recursive structure for the integrations over $W_{k,l}$'s*, which allows to calculate recursively the integrals over $\mathcal{S}$.

Before we start to calculate any of the integrals, for convenience we first distort the triangle in Fig. 1(b) to that of Fig. 8(a). We choose $j_0 = 1$, and relabel the grains in the i-th row as (i, j) with $1 \le j \le i$. Starting at $(k, l) = (N, N)$, we then execute the integrals over $W_{k,l}$ in the decreasing order of both k and l; i.e., for a given value of k, we integrate over $W_{k,l}$ sequentially for decreasing l, we then decrease k by one, and continue the integration process over $W_{k-1,l}$ sequentially for decreasing l, and so on. The integration results over the successive layers are expressed in a "hierarchical nested matrix form" in a recursive manner.

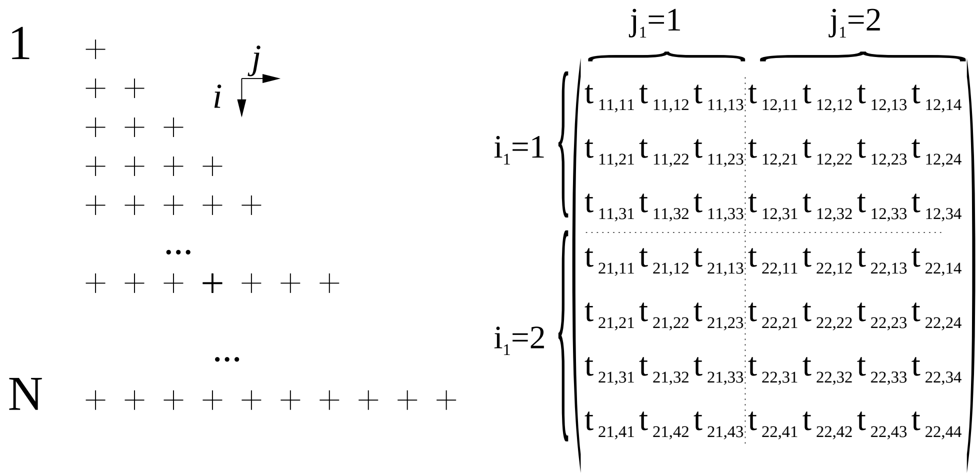

Figure 8. (a) Schematic view of the sublattice we consider; (b) illustration of the "nested" notation used for the matrix elements for $p = 2$.

To illustrate the recursive formulation of the "hierarchical nested matrix form", let us consider the calculation of $\mathcal{N}$. In the expression of $\mathcal{N} = \int_{\mathcal{S}} \prod_{k,l} dW_{k,l}$, evaluating the integrations over $W_{N,N-1}$ through $W_{N,1}$, and $W_{N-1,N-2}$ through $W_{N-1,1}$, we find that the result is a polynomial in $W_{N-2,k}$ that can be written as a matrix product $J_{N-2} = \int \prod_{l=1}^{N-2} dW_{N-1,l} \prod_{k=1}^{N-1} dW_{N,k} = \left[L^{(1)}\right]^{\mathrm{T}} \prod_{l=1}^{N-2} t^{N-1,l} R^{(1)}$. Here, $L^{(1)}$ and $R^{(1)}$ are respectively 1×2 and 2×1 matrices, and the elements of the 2×2 matrix $t^{N-1,l}$ depend only on $W_{N-2,l}$. Thereafter, when J_{N-2} is integrated over $W_{N-2,N-3}$ through $W_{N-1,1}$, the corresponding result $J_{N-3} = \int \prod_{l=1}^{N-3} dW_{N-2,l}\, J_{N-2}$ yields again a polynomial expressible in a similar matrix product form, i.e., $J_{N-3} = \left[L^{(2)}\right]^{\mathrm{T}} \prod_{l=1}^{N-3} t^{N-2,l} R^{(2)}$. The *crucial point to note however is that the matrices* $L^{(2)}$, $t^{N-2,l}$ *and* $R^{(2)}$ *can be constructed by simply unfolding each respective element of* $L^{(1)}$, $t^{N-1,l}$ *and* $R^{(1)}$ *as matrices* [see Fig. 8(b) for the elements of $t^{N-2,l}$, which are now indexed by $i_1 j_1, i_2 j_2$]. After the integration of J_{N-3} over the $W_{N-3,l}$'s, each of the elements of $L^{(2)}$, $t^{N-2,l}$ and $R^{(2)}$ further unfold into matrices and so on. This process continues over further and further integrations generating the "hierarchical nested matrix form". The elements of $L^{(k-1)}$, $t^{N-k+1,l}$ and $R^{(k-1)}$ are related to those of $L^{(k)}$, $t^{N-k,l}$ and $R^{(k)}$ in a recursive manner.

In Sec. 5, we develop the full details of this recursive integration scheme and calculate $\langle W_{i,j} \rangle$ exactly. In Sec. 5.1, we establish some priliminary formulas, which we use over and over again during the course of the exact calculation. In Secs. 5.2 and 5.3 respectively, we evaluate $\mathcal{N}$ and $\int \prod_{k,l} dW_{k,l}\, W_{i,j}$. How to proceed with the calculations of the higher moments and correlations of $W_{i,j}$'s is discussed in Sec. 5.4.

5. Exact Results for $\langle W_{i,j}\rangle$: Details

5.1. Preliminaries

As described in Eqs. (9-11), a full description of a given realization of the contact force configurations is given by the variables $\{q_{i,j}\}_{i=1\ldots N-1,j=1\ldots i}$ with $0 \le q_{i,j} \le 1$. The unnormalized joint probability distribution on these q-variables is $\prod_{i=1}^{N-1}\prod_{j=1}^{i} W_{i,j}$. Here, $W_{1,1} = W_{\rm ext} = 1$, and for $i > 1$, the successive $W_{i,j}$'s are given by

$$\begin{aligned} W_{i,1} &= q_{i-1,1}W_{i-1,1} \\ W_{i,j} &= (1-q_{i-1,j-1})\,W_{i-1,j-1} + q_{i-1,j}\,W_{i-1,j} \qquad \text{for} \quad 1<j<i \text{ and} \\ W_{i,i} &= (1-q_{i-1,i-1})W_{i-1,i-1}\,. \end{aligned} \tag{16}$$

We calculate the moments of $W_{i,j}(q)$ on this joint q-distribution by changing variables from $\{q_{i,j}\}_{i=1\ldots N-1,j=1\ldots i}$ to $\{W_{i,j}\}_{i=2\ldots N,j=\ldots i-1}$. This is achieved by rewriting Eq. (16) as

$$q_{i,j} = \frac{1}{W_{i,j}}\left[\sum_{k=1}^{j} W_{i+1,k} - \sum_{k=1}^{j-1} W_{i,k}\right]. \tag{17}$$

Notice from Fig. 1(b) that since all the forces on the triangle from the left and the right are horizontal, $\sum_{k=1}^{i} W_{i,k} = 1\,\forall i$. This implies that on the i-th layer there are only $(i-1)$ unconstrained $W_{i,j}$'s. We choose them to be $W_{i,j}$ for $1 \le j \le i-1$. Then $W_{i,i} = 1 - \sum_{k=1}^{i-1} W_{i,k}$.

The advantage associated with the change of variables from $\{q_{i,j}\}_{i=1,\ldots,N-1,j=1,\ldots,i}$ to $\{W_{i,j}\}_{i=2,\ldots,N,j=1,\ldots,i-1}$ is that $\prod_{i=1}^{N-1}\prod_{j=1}^{i} dq_{i,j} = \left[\prod_{k=1}^{N-1}\prod_{l=1}^{k} W_{k,l}\right]^{-1}\prod_{i=2}^{N}\prod_{j=1}^{i-1} dW_{i,j}$, so that [from Eq. (11)] $\prod_{ij} dG_{i,j} = \prod_{ij} dW_{i,j}$, i.e., the $W_{i,j}$'s are uniformly distributed. The difficulty of this formulation, however, is that the volume the $W_{i,j}$'s span is not a cube anymore. Instead, the volume spanned is a polygon given by the inequalities

$$a_{i,j} \le W_{i,j} \le b_{i,j}, \tag{18}$$

with

$$\begin{aligned} a_{i,j} &= \sum_{k=1}^{j-1}(W_{i-1,k} - W_{i,k}) \\ b_{i,j} &= W_{i-1,j} + \sum_{k=1}^{j-1}(W_{i-1,k} - W_{i,k}) \end{aligned} \tag{19}$$

The $a_{i,j}$'s and the $b_{i,j}$'s are related by the following simple relations:

$$b_{i,j} - a_{i,j} = W_{i-1,j} \tag{20}$$

$$a_{i,j} = b_{i,j-1} - W_{i,j-1} \tag{21}$$

$$b_{i,1} = W_{i-1,1} \tag{22}$$

$$a_{i,1} = 0\,. \tag{23}$$

Finally, the following integral is the centrepiece of all our calculations:

$$I_{mn}(a,b) = \int_a^b x^m (b-x)^n dx = \sum_{k=0}^{m} \alpha_{mn}^{m-k}\,(b-a)^{n+1+k}\,a^{m-k}\,, \tag{24}$$

where

$$\alpha_{mn}^{m-k} = \frac{m!\,n!}{(n+k+1)!\,(m-k)!}\,. \tag{25}$$

5.2. Evaluation of the Normalization constant $\mathcal{N}$

In the $W_{i,j}$-space, the average of a quantity h is given by $\langle h\rangle = \left[\int\prod_{ij} dW_{i,j}h\right]/\mathcal{N}$, where $\mathcal{N} = \int\prod_{ij} dW_{i,j}$ is the normalization constant. In Sec. 5.2, we evaluate $\mathcal{N}$ by performing the integrations over $\prod_{ij} W_{i,j}$ layer by layer bottom up; i.e., we start at $i = N$ and decrease i until we reach the top layer where $i = 1$. At each layer, the integrations are carried out one by one in the direction of decreasing j, from $j = i-1$ to $j = 1$. The integral $J_{N-p} = \int\prod_{i=N-p+1}^{N}\prod_{j=1,\ldots,i-1} dW_{i,j}$ is the unnormalized induced probability density of the $W_{i,j}$'s for the $(N-p)$ top layers. In this notation, $\mathcal{N} = J_1$.

Our exact calculations are made possible due to the particular forms of the bounds $a_{i,j}$'s and $b_{i,j}$'s, as they introduce a recursive structure. We will evaluate J_{N-1} and J_{N-2} explicitly, after which we will prove the general recurrence relation for J_{N-p} by induction. In fact, we will show that J_{N-p} can be written as a matrix product. The matrices entering this product are given by recursive relations, which we will call the "fundamental relations", as they are the building blocks of the calculation of all moments of $W_{i,j}$.

5.2.1. Evaluation of J_{N-1} and J_{N-2}: By definition, $J_{N-1} = \int\prod_{j=1}^{N-1} dW_{N,j}$. We integrate the $W_{N,j}$'s one by one in the direction of decreasing j's. Using

$$\int_{a_{N,j-1}}^{b_{N,j-1}} dW_{N,j-1} = b_{N,j-1} - a_{N,j-1} = W_{N-1,N-1}\,, \tag{26}$$

we have

$$J_{N-1} = \int\prod_{j=1}^{N-1} dW_{N,j} = \prod_{j=1}^{N-1} W_{N-1,j}\,.$$

In order to obtain J_{N-2}, we have to integrate J_{N-1} w.r.t. $W_{N-1,j}$ for $1\le j\le N-2$. Since the bounds $a_{i,j}$ and $b_{i,j}$ depend only on $W_{i,l}$ and $W_{i-1,l}$ for $l<j$, J_{N-2} can be expressed in a nested form:

$$\begin{aligned} J_{N-2} &= \int\left[\prod_{j=1}^{N-2} dW_{N-1,j}\right] J_{N-1} \\ &= \int_{a_{N-1,1}}^{b_{N-1,1}} dW_{N-1,1}W_{N-1,1}\ldots\int_{a_{N-1,N-2}}^{b_{N-1,N-2}} dW_{N-1,N-2}\,W_{N-1,N-2}\,W_{N-1,N-1}\,, \end{aligned} \tag{27}$$

and thereafter it can be evaluated iteratively. Having defined $J^{N-2,N-1} = W_{N-1,N-1} = 1-\sum_{k=1}^{N-2} W_{N-1,k}$, we have

$$J^{N-2,k} = \int_{a_{N-1,k}}^{b_{N-1,k}} dW_{N-1,k}\,W_{N-1,k}\,J^{N-2,k+1} \tag{28}$$

for $k > 1$, i.e., $J_{N-2} = J^{N-2,1}$. We now show that $\forall\, k \geq 1$, $J^{N-2,k}$ is of the form $\gamma_1^{N-1,k-1}(b_{N-1,k-1} - W_{N-1,k-1}) + \gamma_2^{N-1,k-1}$, where the vectors $\gamma^{N-1,k-1} = \begin{bmatrix} \gamma_1^{N-1,k-1} \\ \gamma_2^{N-1,k-1} \end{bmatrix}$ and $\gamma^{N-1,k} = \begin{bmatrix} \gamma_1^{N-1,k} \\ \gamma_2^{N-1,k} \end{bmatrix}$ are related by means of a matrix relation of the form $\gamma^{N-1,k-1} = t^{N-1,k}\,\gamma^{N-1,k}$.

For $k = N-1$, the proposed form of $J^{N-2,k}$ is easily checked by using $\sum_{k=0}^{N-2} W_{N-2,k} = 1$ and Eq. (18):

$$\begin{aligned} J^{N-2,N-1} &= 1 - \sum_{k=1}^{N-2} W_{N-1,k} = \sum_{k=1}^{N-2} \left[W_{N-2,k} - W_{N-1,k}\right] = b_{N-1,N-2} - W_{N-1,N-2} \\ &= \gamma_1^{N-1,N-2}\left[b_{N-1,N-2} - W_{N-1,N-2}\right] + \gamma_2^{N-1,N-2}\,, \end{aligned} \tag{29}$$

with $\gamma_1^{N-1,N-2} = 1$ and $\gamma_2^{N-1,N-2} = 0$. This means that $\gamma^{N-1,N-2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv R^{(1)}$.

Let us now assume that the form $J^{N-2,k+1} = \gamma_1^{N-1,k}(b_{N-1,k} - W_{N-1,k}) + \gamma_2^{N-1,k}$ holds for a given k, $1 < k < N-2$. Then from Eqs. (24) and (28), we have

$$\begin{aligned} J^{N-2,k} &= \gamma_1^{N-1,k} I_{11}(a_{N-1,k}, b_{N-1,k}) + \gamma_2^{N-1,k} I_{10}(a_{N-1,k}, b_{N-1,k}) \\ &= \left[\gamma_1^{N-1,k}\alpha_{11}^1 (b_{N-1,k} - a_{N-1,k})^2 + \gamma_2^{N-1,k}\alpha_{10}^1 (b_{N-1,k} - a_{N-1,k})\right] a_{N-1,k} \\ &\quad + \left[\gamma_1^{N-1,k}\alpha_{11}^0 (b_{N-1,k} - a_{N-1,k})^3 + \gamma_2^{N-1,k}\alpha_{10}^0 (b_{N-1,k} - a_{N-1,k})^2\right]. \end{aligned} \tag{30}$$

Thereafter, using Eqs. (20) and (21), $J^{N-2,k}$ can be rewritten as

$$J^{N-2,k} = \gamma_1^{N-1,k-1}(b_{N-1,k-1} - W_{N-1,k-1}) + \gamma_2^{N-1,k-1}\,, \tag{31}$$

with

$$\begin{aligned} \gamma_1^{N-1,k-1} &= \gamma_1^{N-1,k}\alpha_{11}^1 W_{N-2,k}^2 + \gamma_2^{N-1,k}\alpha_{10}^1 W_{N-2,k} \\ \gamma_2^{N-1,k-1} &= \gamma_1^{N-1,k}\alpha_{11}^0 W_{N-2,k}^3 + \gamma_2^{N-1,k}\alpha_{10}^0 W_{N-2,k}^2\,. \end{aligned} \tag{32}$$

In other words, we have the matrix relation $\gamma^{N-1,k-1} = t^{N-1,k}\,\gamma^{N-1,k}$, where $t^{N-1,k}$ is a 2×2 matrix with elements

$$t_{i_1 j_1}^{N-1,k} = \alpha_{1,2-j_1}^{2-i_1}\, W_{N-2,k}^{2+i_1-j_1}\,. \tag{33}$$

Thus, by means of the induction procedure via Eqs. (30-33), we have demonstrated that $J^{N-2,2} = \prod_{k=2}^{N-2} t^{N-1,k}\, R^{(1)}$. Finally, in the last integral (over $W_{N-1,1}$) in Eq. (27), the lower limit $a_{N-1,1} = 0$, and it is easily seen that

$$J_{N-2} = \int_{a_{N-1,1}}^{b_{N-1,1}} dW_{N-1,1}\, W_{N-1,1} J^{N-2,2} = \left[L^{(1)}\right]^{\mathrm{T}} J^{N-2,1} = \left[L^{(1)}\right]^{\mathrm{T}} \prod_{k=1}^{N-2} t^{N-1,k}\, R^{(1)}, \tag{34}$$

where $\left[L^{(1)}\right]^{\mathrm{T}}$ is the row vector [0 1].

5.2.2. Expression of J_{N-p} by induction: We now derive the general formula for J_{N-p} by induction. The postulate is that the induced probability density of the $N-p$ top layers can be written as

$$J_{N-p} = \left[L^{(p-1)}\right]^{\mathrm{T}} \prod_{k=1}^{N-p} t^{N-(p-1),k}\, R^{(p-1)}\,, \tag{35}$$

where $t^{N-(p-1),k}$, expressed in "hierarchical nested matrix form", has elements

$$t^{N-(p-1),k}_{i_1j_1,i_2j_2,\ldots,i_{p-1}j_{p-1}} = \beta_{i_1j_1,i_2j_2,\ldots,i_{p-1}j_{p-1}} W^{p+i_{p-1}-j_{p-1}}_{N-p,k} \tag{36}$$

with

$$\beta_{i_1j_1,\ldots,i_{p-1}j_{p-1}} = \alpha^{2-i_1}_{1,2-j_1} \prod_{l=2}^{p-1} \alpha^{l+i_{l-1}-i_l}_{l+i_{l-1}-j_{l-1},l+j_{l-1}-j_l} \Theta(i_l - j_{l-1})\,. \tag{37}$$

Here $\Theta(k)$ is the discrete Heavyside Step-function, i.e., $\Theta(k) = 1$ if $k \geq 0$, else $\Theta(k) = 0$. An alternative recursive formulation for the β's is given by

$$\begin{aligned} \beta_{i_1j_1} &= \alpha^{2-i_1}_{1,2-j_1} \qquad \text{and} \\ \beta_{i_1j_1,\ldots,i_pj_p} &= \beta_{i_1j_1,\ldots,i_{p-1}j_{p-1}}\, \alpha^{p+i_{p-1}-i_p}_{p+i_{p-1}-j_{p-1},p+j_{p-1}-j_p} \Theta(i_p - j_{p-1}) \quad \forall p > 1. \end{aligned} \tag{38}$$

The expressions (36-38) will constantly be referred to in all the following calculations, and we call them the "fundamental relations".

The vectors $L^{(p-1)}$ and $R^{(p-1)}$, respectively, are

$$\begin{aligned} L^{(p-1)}_{i_1,\ldots,i_{p-1}} &= \prod_{l=1}^{p-1} \delta_{i_l,\frac{l(l+1)}{2}+1} \qquad \text{and} \\ R^{(p-1)}_{i_1,\ldots,i_{p-1}} &= \delta_{j_{p-1},1} \sum_{j_1\cdots j_{p-2}} \beta_{i_1j_1,\ldots,i_{p-2}j_{p-2}} R^{(p-2)}_{j_1,\ldots,j_{p-2}}\,. \end{aligned} \tag{39}$$

"Hierarchical nested matrix form" means that elements are referenced through nested blocks. For example, the element referenced by $i_1j_1, i_2j_2, \ldots, i_{p-1}j_{p-1}$ is the $i_{p-1}j_{p-1}$-th sub-block of the block $i_1j_1, i_2j_2, \ldots, i_{p-2}j_{p-2}$, which itself is the $(i_{p-1}j_{p-1})$-th sub-block of the block $i_1j_1, i_2j_2, \ldots, i_{p-3}j_{p-3}$, and so on [see Fig. 8(b) for an illustration in the case $p = 2$]. The indices vary between the following bounds:

$$\begin{aligned} &1 \leq i_1 \leq 2, && 1 \leq j_1 \leq 2 \\ &1 \leq i_2 \leq 2 + i_1, && 1 \leq j_2 \leq 2 + j_1 \\ &\quad\vdots \\ &1 \leq i_l \leq l + i_{l-1}, && 1 \leq j_l \leq l + j_{l-1} \\ &\quad\vdots \\ &1 \leq i_{p-1} \leq p - 1 + i_{p-2}, && 1 \leq j_{p-1} \leq p - 1 + j_{p-2}\,. \end{aligned} \tag{40}$$

In our induction procedure, we assume the above forms (35-40) for a given p, and then show that these relations also hold when p is replaced by $p+1$. The two step route for this induction procedure that we follow is the same as the one we followed to evaluate J_{N-2}.

Step (i): We first write

$$J_{N-p-1} = \int \left[\prod_{k'=1}^{N-p-1} dW_{N-p,k'} \right] \left[L^{(p-1)}\right]^{\mathrm{T}} \prod_{k=1}^{N-p} t^{N-(p-1),k} R^{(p-1)}\,, \tag{41}$$

which we will evaluate in an iterative manner as we did in Sec. 5.2.1. We define the vector $J^{N-p-1,N-p} = t^{N-(p-1),N-p} R^{(p-1)}$ for $1 < k < N-p$, and the successive integrals

are iteratively given by

$$J^{N-p-1,k} = \int_{a_{N-p,k}}^{b_{N-p,k}} dW_{N-p,k}\, t^{N-(p-1),k}\, J^{N-p-1,k+1}\,. \tag{42}$$

The integral sign in Eq. (42) is interpreted in the sense that we integrate each component of the vector integrand. The final integral (over $W_{N-p,1}$) yields, just as we saw in Sec. 5.2.1, $J_{N-p-1} = \left[L^{(p-1)}\right]^{\mathrm{T}} J^{N-p-1,1}$.

Step (ii): We then show that the $(i_1, i_2, \ldots, i_{p-1})$-th component of $J^{N-p-1,k}$ is given by

$$J^{N-p-1,k}_{i_1,\ldots,i_{p-1}} = \sum_{i_p=1}^{p+i_{p-1}} \gamma^{N-p,k-1}_{i_1,\ldots,i_p}\, (b_{N-p,k-1} - W_{N-p,k-1})^{p+j_{p-1}-j_p}. \tag{43}$$

The vectors $\gamma^{N-p,k-1}$ and $\gamma^{N-p,k}$ with elements $\gamma^{N-p,k-1}_{i_1,\ldots,i_p}$ and $\gamma^{N-p,k}_{i_1,\ldots,i_p}$ respectively are related to each other via the matrix relation $\gamma^{N-p,k-1} = t^{N-p,k}\,\gamma^{N-p,k}$.

Starting with $J^{N-p-1,N-p}$, we have

$$\begin{aligned} J^{N-p-1,N-p}_{j1,\ldots,j_{p-1}} &= \sum_{k_1,k_2,\ldots,k_{p-1}} \beta_{j_1k_1,j_2k_2,\ldots,j_{p-1}k_{p-1}} W^{p+j_{p-1}-k_{p-1}}_{N-p,N-p}\, R^{p-1}_{k_1,k_2,\ldots,k_{p-1}} \\ &= \sum_{k_1,k_2,\ldots,k_{p-1}} \beta_{j_1k_1,j_2k_2,\ldots,j_{p-1}k_{p-1}} W^{p+j_{p-1}-1}_{N-p,N-p} \delta_{k_{p-1},1} R^{(p-1)}_{k_1,k_2,\ldots,k_{p-2}} \\ &= \sum_{j_p=1}^{p+j_{p-1}} \gamma^{N-p,N-p}_{j_1,j_2,\ldots,j_p}\, (b_{N-p,N-p-1} - W_{N-p,N-p-1})^{p+j_{p-1}-j_p}\,, \end{aligned} \tag{44}$$

where we have used Eq. (39) between the first and the second lines of Eq. (44), and the fact that $W_{N-p,N-p} = b_{N-p,N-p-1} - W_{N-p,N-p-1}$ between the second and the third lines. In other words, $J^{N-p-1,N-p}$ indeed has the postulated form (43 with $\gamma^{N-p,N-p}_{j_1,\ldots,j_p} = \delta_{j_p,1} \sum_{k_1,\ldots,k_{p-1}} \beta_{j_1k_1,\ldots,j_{p-1}k_{p-1}} R^{(p-1)}_{k1,\ldots,k_{p-2}}$, where we choose j_p to vary between 1 and $p + j_{p-1}$, as shown in Eq. (40).

Assuming the form (43) for $J^{N-p-1,k}$, we now calculate $J^{N-p-1,k-1}$ using Eq. (42). The $(i_1, \ldots, i_{p-1})$-th element of the corresponding integrand is

$$\sum_{j_1,\ldots,j_p} \beta_{i_1j_1,\ldots,i_{p-1}j_{p-1}} W^{p+i_{p-1}-j_{p-1}}_{N-p,k-1} \gamma^{N-p,k-1}_{j_1,\ldots,j_p} (b_{N-p,k-1} - W_{N-p,k-1})^{p+j_{p-1}-j_p}\,, \tag{45}$$

so that after using Eq. (24), we obtain

$$J^{N-p-1,k-1}_{i_1,\ldots,i_{p-1}} = \sum_{j_1,\ldots,j_p} \beta_{i_1j_1,\ldots,i_{p-1}j_{p-1}} \gamma^{N-p,k-1}_{j_1,\ldots,j_p}\, I_{p+i_{p-1}-j_{p-1},p+j_{p-1}-j_p}(a_{N-p,k-1}, b_{N-p,k-1}). \tag{46}$$

Using Eqs. (20) and (24), we now expand $I_{p+i_{p-1}-j_{p-1},p+j_{p-1}-j_p}(a_{N-p,k-1}, b_{N-p,k-1})$ as

$$\begin{aligned} &I_{p+i_{p-1}-j_{p-1},p+j_{p-1}-j_p}(a_{N-p,k-1}, b_{N-p,k-1}) \\ &\quad = \sum_{l=0}^{p+i_{p-1}-j_{p-1}} \alpha^{p+i_{p-1}-j_{p-1}-l}_{p+i_{p-1}-j_{p-1},p+j_{p-1}-j_p}\, a^{p+i_{p-1}-j_{p-1}-l}_{N-p,k-1}\, W^{p+1+j_{p-1}+l-j_p}_{N-p-1,k-1} \\ &\quad = \sum_{i_p=1}^{p+i_{p-1}} \Theta(i_p - j_{p-1}) \alpha^{p+i_{p-1}-i_p}_{p+i_{p-1}-j_{p-1},p+j_{p-1}-j_p}\, a^{p+i_{p-1}-i_p}_{N-p,k-1}\, W^{p+1+i_p-j_p}_{N-p-1,k-1}\,, \end{aligned} \tag{47}$$

where in the last line of Eq. (47), $i_p = j_{p-1} + l$. Thereafter, we rewrite Eq. (46) using Eq. (47) as

$$\begin{aligned}
J^{N-p-1,k-1}_{i_1,\dots,i_{p-1}} &= \sum_{j_1,\dots,j_p} \beta_{i_1 j_1,\dots,i_{p-1}j_{p-1}} \gamma^{N-p,k-1}_{j_1,\dots,j_p} \sum_{i_p=1}^{p+i_{p-1}} \Theta(i_p - j_{p-1}) \times \\
&\qquad \times \alpha^{p+i_{p-1}-i_p}_{p+i_{p-1}-j_{p-1},p+j_{p-1}-j_p} a^{p+i_{p-1}-i_p}_{N-p,k-1} W^{p+1+i_p-j_p}_{N-p-1,k-1} \\
&= \sum_{i_p=1}^{p+i_{p-1}} \gamma^{N-p,k-2}_{i_1,\dots,i_p} a^{p+i_{p-1}-i_p}_{N-p,k-1} = \sum_{i_p}^{p+i_{p-1}} \gamma^{N-p,k-2}_{i_1,\dots,i_p} (b_{N-p,k-2} - W_{N-p,k-2})^{p+j_{p-1}-j_p} \quad (48)
\end{aligned}$$

to recover

$$\gamma^{N-p,k-2}_{i_1,\dots,i_p} = \sum_{j_1,\dots,j_p} t^{N-p,k-1}_{i_1 j_1,\dots,i_p j_p} \gamma^{N-p,k-1}_{j_1,\dots,j_p} \quad (49)$$

and

$$\begin{aligned}
t^{N-p,k-1}_{i_1 j_1,\dots,i_p j_p} &= \beta_{i_1 j_1,\dots,i_{p-1}j_{p-1}} \alpha^{p+i_{p-1}-i_p}_{p+i_{p-1}-j_{p-1},p+j_{p-1}-j_p} \Theta(i_p - j_{p-1}) W^{p+1+i_p-j_p}_{N-p-1,k-1} \\
&= \alpha^{2-i_1}_{1,2-j_1} \prod_{l=2}^{p} \alpha^{l+i_{l-1}-i_l}_{l+i_{l-1}-j_{l-1},l+j_{l-1}-j_l} \Theta(i_l - j_{l-1}) W^{p+1+i_p-j_p}_{N-p-1,k-1} . \quad (50)
\end{aligned}$$

Notice that $t^{N-p,k-1}$ indeed has the form postulated by Eq. (36).

This procedure outlined in Eqs. (45-50) allows us to evaluate all the integrals in Eq. (41) up to $k = 1$. Each iteration in Eq. (41) introduces an extra multiplicative factor of the matrix $t^{N-p,k}$, whose form is given in Eq. (50).

For the last integration (over $W_{N-p,1}$) in Eq. (41), $a_{N-p,1} = 0$ and $b_{N-p,1} = 1$, so that $J^{N-p-1,1}_{i_1,\dots,i_{p-1}}$ is simply equal to $\gamma^{N-p,1}_{i_1,i_2,\dots,i_{p-1},p+i_{p-1}}$; i.e.,

$$\begin{aligned}
J_{N-p-1} &= \sum_{i_1,i_2,\dots i_{p-1}} L^{(p-1)}_{i_1,i_2,\dots,i_{p-1}} \gamma^{N-p,1}_{i_1,i_2,\dots,i_{p-1},p+i_{p-1}} \\
&= \sum_{i_1,i_2,\dots,i_p} L^{p-1}_{i_1,i_2,\dots,i_{p-1}} \delta_{i_p,p+i_{p-1}} \gamma^{N-p,1}_{i_1,i_2,\dots,i_{p-1},i_p} \\
&= \sum_{i_1,i_2,\dots,i_p} L^{(p-1)}_{i_1,i_2,\dots,i_{p-1}} \delta_{i_p,p+i_{p-1}} \left[\prod_{k=1}^{N-p-1} t^{N-p,k} \gamma^{N-p,N-p} \right]_{i_1,i_2,\dots,i_{p-1},i_p} \\
&= \left[L^{(p)} \right]^{\mathrm{T}} \prod_{k=1}^{N-p-1} t^{N-p,k} R^{(p)} , \quad (51)
\end{aligned}$$

where

$$L^{(p)}_{i_1,\dots,i_p} = L^{(p-1)}_{i_1,\dots,i_{p-1}} \delta_{i_p,p+i_{p-1}} = \prod_{l=1}^{p} \delta_{i_l,\frac{l(l+1)}{2}+1} \quad (52)$$

and

$$R^{(p)}_{j_1,j_2,\dots,j_p} = \gamma^{N-p,N-p}_{j_1,j_2,\dots,j_p} = \delta_{j_p,1} \sum_{k_1,k_2,\dots,k_{p-1}} \beta_{j_1 k_1,j_2 k_2,\dots,j_{p-1}k_{p-1}} R^{(p-1)}_{k_1,\dots,k_{p-1}} ; \quad (53)$$

i.e., Eqs. (52-53) are exactly of the form postulated in Eq. (39).

5.2.3. The normalization constant $\mathcal{N}$: The above induction procedure allows us to evaluate J_{N-p} all the way to $p = N - 1$, and the expression for the normalization constant is then given by

$$\mathcal{N} = J_1 = \left[L^{(N-2)}\right]^{\mathrm{T}} t^{2,1}\, R^{(N-2)} . \tag{54}$$

For the full expression of $\mathcal{N}$ however, we also need to use Eqs. (25), (36) and (37) to obtain

$$t^{2,1}_{i_1 j_1, i_2 j_2, \ldots, i_{N-2} j_{N-2}} = \beta_{i_1 j_1, i_2 j_2, \ldots, i_{N-2} j_{N-2}} \qquad \text{since } W_{1,1} = W_{\text{ext}} = 1\,, \tag{55}$$

$$\beta_{i_1 j_1,\ldots,i_p j_p} = \frac{(2-j_1)!}{(2-i_1)!}\frac{(2+j_1-j_2)!}{(2+i_1-i_2)!}\cdots\frac{(p+j_{p-1}-j_p)!}{(p+i_{p-1}-i_p)!}\,\frac{1}{(p+1+i_p-j_p)!}\prod_{l=2}^{p-1}\Theta(i_l-j_{l-1}), \tag{56}$$

and further Eqs. (25) and (39) to derive

$$R^{(p)}_{i_1,\ldots,i_p} = \frac{\Lambda^{(p)}_{i_1\ldots,i_p}}{(2-i_1)!(2+i_1-i_2)!\ldots(p-1+i_{p-2}-i_{p-1})!(p-1+i_{p-1})!} . \tag{57}$$

In Eq. (57), $\Lambda^{(p)}_{j_1 \ldots j_p}$ are integers given by the following recursive formula:

$$\begin{aligned} \Lambda^{(2)}_{j_1 j_2} &= \delta_{j_2,1} \\ \Lambda^{(p+1)}_{j_1,\ldots,j_{p+1}} &= \delta_{j_{p+1},1} \sum_{k_1,\ldots,k_p} \Lambda^{(p)}_{k_1,\ldots,k_p} \prod_{l=1}^{p} \Theta(j_{l+1} - k_l) . \end{aligned} \tag{58}$$

Finally, using Eqs. (54-58) and (52), we have

$$\mathcal{N} = J_1 = \frac{1}{\left[\frac{(N-1)N}{2}\right]!} \sum_{j_1,\ldots,j_{N-2}} \Lambda^{(N-2)}_{j_1,\ldots,j_{N-2}} . \tag{59}$$

5.3. Calculation of the Expectation Values $\langle W_{N-q,r}\rangle$

5.3.1. Modifications of fundamental relations: In this section, we obtain the general form of $\langle W_{N-q,r}\rangle$, the mean value of $W_{N-q,r}$ at q layers from the bottom and r grains from the left boundary [see Fig. 9 for illustration], for $0 \le q \le N-1$ and $1 \le r \le N-q-1$, in four steps. From what we learnt in Sec. 5.2, it is clear that in order to calculate $\langle W_{N-q,r}\rangle$, we need to integrate $W_{N-q,r}J_{N-q}$ from the $(N-q)$-th layer to all the way to the top. Just like we saw in Sec. 5.2, these integrations are equivalent to matrix multiplications layer by layer in an iterative manner, but the fundamental relations (36-38) do get slightly modified.

Step (i): We start with the calculation of the integral of $W_{N-q,r}J_{N-q}$ on the $(N-q)$-th layer

$$\bar{J}_{N-q-1}(q,r) = \int \left[\prod_{l'=1}^{N-q-1} dW_{N-q,l'}\right] W_{N-q,r}\left[L^{(q-1)}\right]^{\mathrm{T}} \left[\prod_{l=1}^{N-q} t^{N-(q-1),l}\right] R^{(q-1)}, \tag{60}$$

wherein once again we carry out the integrations iteratively from $l' = N-q-1$ down to $l' = 1$. Of them, notice that the integrations for $l' = N-q-1$ down to $l' = r+1$

proceed exactly as in Eqs. (43-50), yielding $\bar{J}^{N-q-1,r+1}(q,r) \equiv J^{N-q-1,r+1}$, i.e.,

$$\begin{aligned}\bar{J}_{N-q-1}(q,r) = &\int \left[\prod_{l'=1}^{r} dW_{N-q,l'}\right] W_{N-q,r} \left[L^{(q-1)}\right]^{\mathrm{T}} \left[\prod_{l=1}^{r} t^{N-(q-1),l}\right] J^{N-q-1,r+1}(q,r) \\ \equiv &\int \left[\prod_{l'=1}^{r-1} dW_{N-q,l'}\right] \left[L^{(q-1)}\right]^{\mathrm{T}} \left[\prod_{l=1}^{r-1} t^{N-(q-1),l}\right] \bar{J}^{N-q-1,r}(q,r)\,. \end{aligned} \tag{61}$$

Step (ii): For $l'=r$, the recursive integration procedure differs from those in Eqs. (43-50) due to the presence of an extra factor of $W_{N-q,r}$ in the integrand, and thus the integration over $W_{N-q,r}$ in Eq. (60) requires a slightly different treatment.We first write $\bar{J}^{N-q-1,r}(q,r)$ explicitly:

$$\begin{aligned}\bar{J}^{N-q-1,r}_{i_1,\ldots,i_{q-1}}(q,r) = &\int_{a_{N-q,r}}^{b_{N-q,r}} dW_{N-q,r} W_{N-q,r} \sum_{j_1,\ldots,j_q} \beta_{i_1 j_1,\ldots,i_{q-1}j_{q-1}} W_{N-q,r}^{q+i_{q-1}-j_{q-1}} \times \\ &\times \gamma^{N-q,r+1}_{j_1,\ldots,j_q} (b_{N-q,r} - W_{N-q,r})^{q+j_{q-1}-j_q} \\ = &\sum_{j_1,\ldots,j_q} \beta_{i_1 j_1,\ldots,i_{q-1}j_{q-1}} \gamma^{N-q,r+1}_{j_1,\ldots,j_q} I_{q+1+i_{q-1}-j_{q-1},q+j_{q-1}-j_q}(a_{N-q,r}, b_{N-q,r})\,. \end{aligned} \tag{62}$$

We then use

$$\begin{aligned} &I_{q+1+i_{q-1}-j_{q-1},q+j_{q-1}-j_q}(a_{N-q,r}, b_{N-q,r}) \\ &= \sum_{l=0}^{q+1+i_{q-1}-j_{q-1}} \alpha^{q+1+i_{q-1}-j_{q-1}-l}_{q+1+i_{q-1}-j_{q-1},q+j_{q-1}-j_q}\, a_{N-q,r}^{q+1+i_{q-1}-j_{q-1}-l}\, W_{N-q-1,r}^{q+1+j_{q-1}+l-j_q} \\ &= \sum_{i_q=1}^{q+1+i_{q-1}} \Theta(i_q - j_{q-1})\, \alpha^{q+1+i_{q-1}-i_q}_{q+1+i_{q-1}-j_{q-1},q+j_{q-1}-j_q}\, a_{N-q,r}^{q+1+i_{q-1}-i_q}\, W_{N-q-1,r}^{q+1+i_q-j_q} \end{aligned} \tag{63}$$

(where once again we introduce $i_q = j_{q-1} + l$), to get

$$\bar{J}^{N-q-1,r}_{i_1,\ldots,i_{q-1}}(q,r) = \sum_{i_q=1}^{q+1+i_{q-1}} \bar{\gamma}^{N-q,r}_{i_1,\ldots,i_q} \left(b_{N-q,r-1} - W_{N-q,r-1}\right)^{q+1+i_{q-1}-i_q} \tag{64}$$

with $\bar{\gamma}^{N-q,r} = \bar{t}(q,r)^{N-q,r}\, \gamma^{N-q,r+1}$, where

$$\begin{aligned}\bar{t}^{N-q,r}_{i_1 j_1,\ldots,i_q j_q}(q,r) &= \beta_{i_1 j_1,\ldots,i_{q-1}j_{q-1}} \Theta(i_q - j_{q-1})\, \alpha^{q+1+i_{q-1}-i_q}_{q+1+i_{q-1}-j_{q-1},q+j_{q-1}-j_q}\, W_{N-q-1,r}^{q+1+i_q-j_q} \\ &= \bar{\beta}^{q,r}_{i_1 j_1,\ldots,i_q j_q}\, W_{N-q-1,r}^{q+1+i_q-j_q}\,. \end{aligned} \tag{65}$$

It is worthwhile to note that all indices except i_q in Eqs. (63-65) run between the bounds defined in (40), while for i_q, we have $1 \le i_q \le q+1+i_{q-1}$. In other words, the fundamental relation (36) gets modified, as $\bar{t}^{N-q,r}$ has one more row than $t^{N-q,r}$. Such a modification can be thought of as a "defect" in the recursive integration procedure (43-50). More precisely, we call Eq. (65) a "defect of type I at (q,r)" and it is characterized by the following relation between β and $\bar{\beta}^{q,r}$ [this relation is obtained via the usage of Eq. (25)]:

$$\bar{\beta}^{q,r}_{i_1 j_1,\ldots,i_q j_q} = \beta_{i_1 j_1,\ldots,i_q j_q} \frac{q+1+i_{q-1}-j_{q-1}}{q+1+i_{q-1}-i_q}\,. \tag{66}$$

Step (iii): Next we show that one needs yet another recursive form when $l' = r-1$ in Eq. (60), but thereafter from $l' = r-2$ down to $l' = 1$, the recursive integration scheme of Eq. (60) returns to its old form (43-50).

The recursive form for $l' = r-1$ is easily obtained by taking the matrix product of $t^{N-(q-1),r-1}$ and $\bar{J}^{N-q-1,r}(q,r)$ and then integrating it w.r.t. $W_{N-q,r-1}$, yielding

$$\begin{aligned}\bar{J}^{N-q-1,r-1}_{i_1\ldots i_{q-1}}(q,r) &= \sum_{j_1,\ldots,j_q} \beta_{i_1j_1,\ldots,i_{q-1}j_{q-1}}\, \bar{\gamma}^{N-q,r}_{j_1,\ldots,j_q}\, I_{q+i_{q-1}-j_{q-1},q+1+j_{q-1}-j_q}(a_{N-q,r-1},b_{N-q,r-1})\\ &= \sum_{i_q=1}^{q+i_{q-1}} \bar{\gamma}^{N-q,r-1}_{i_1,\ldots,i_q}\,(b_{N-q,r-2}-W_{N-q,r-2})^{q+i_{q-1}-i_q}\,,\end{aligned} \tag{67}$$

where the last line of Eq. (67) has been obtained by using Eq. (24). Thereafter, using Eq. (25), we get $\bar{\gamma}^{N-q,r-1} = \bar{t}^{N-q,r-1}(q,r)\bar{\gamma}^{N-q,r}$, with

$$\bar{t}^{N-q,r-1}_{i_1j_1,\ldots,i_qj_q}(q,r) = \beta_{i_1j_1,\ldots,i_{q-1}j_{q-1}}\,\Theta(i_q-j_{q-1})\,\alpha^{q+i_{q-1}-i_q}_{q+i_{q-1}-j_{q-1},q+1+j_{q-1}-j_q}\,W^{q+2+i_q-j_q}_{N-q-1,r-1}\,. \tag{68}$$

In Eq. (68), all indices except j_q satisfy Eq. (40), and for j_q, we have $1\le j_q\le q+1+j_{q-1}$; i.e., the fundamental relation is once again modified: $\bar{t}^{N-q,r-1}(q,r)$ has one more column than $t^{N-q,r-1}$. We call this modification a "defect of type II at $(q,r-1)$", which is characterized by the following relation between β and $\bar{\beta}^{q,r-1}$ [once again, obtained via the usage of Eq. (25)]:

$$\bar{\beta}^{q,r-1}_{i_1j_1,i_2j_2,\ldots i_qj_q} = \beta_{i_1j_1,i_2j_2,\ldots i_qj_q}\,\frac{q+1+j_{q-1}-j_q}{q+2+i_q-j_q}\,. \tag{69}$$

As for the integrations in Eq. (60) for $l' = r-2$ down to $l' = 1$, notice that the final expression (67) is the same as the earlier expression (44), although the matrices relating the γ's have been modified. Thus, the rest of the integrations in Eq. (60) for the $(N-q)$-th layer yield the recurrence relation (43), and the fundamental relations for the matrices remain the same as in Eq. (36). The final result then simply becomes

$$\bar{J}_{N-q-1} = \left[L^{(q)}\right]^{\mathrm{T}} \prod_{l'=1}^{r-2} t^{N-q,l'}\,\bar{t}^{N-q,r-1}(q,r)\,\bar{t}^{N-q,r}(q,r) \prod_{l=r+1}^{N-q-1} t^{N-q,l} R^{(q)}, \tag{70}$$

i.e., in the $(N-q)$-th layer, only two of the matrices get modified w.r.t. Eq. (36).

Step (iv): In the fourth step, we integrate $\bar{J}_{N-q-1}$ for the remaining $N-q-2$ layers to the top of the pile one by one. This integration procedure is the same as what has been detailed in steps (i)-(iii), so here we only provide a short description of it.

When we integrate $\bar{J}_{N-q-1}$ in the $(N-q-1)$-th layer, it is easily seen [following the calculations in steps (i)-(iii) above] that the fundamental relations for the matrices remain the same as in Eq. (36) for the integrations over $W_{N-q-1,N-q-2}$ down to $W_{N-q-1,r}$. Two defects, one of type I and another of type II appear respectively at locations $(N-q-1,r-1)$ and at $(N-q-1,r-2)$, but the fundamental relations (36) are recovered for the integrations over $W_{N-q-1,r-3}$ down to $W_{N-q-1,1}$. In other words, after the integrations of $\bar{J}_{N-q-1}$ over all the $W_{N-q-1,j}$'s for $j=(N-q-2),\ldots,1$, the locations of the defects move one grain each towards the left. In fact, this trend of the leftward shift of the defects by one grain each time we integrate over all the $W_{i,j}$'s in the successive layer upwards continues to hold until the locations of the defects reach (and terminate on) the left boundary [see Fig. 9].

In summary, effectively, the difference between the calculations of $\mathcal{N}$ and that of $\langle W_{N-q,r}\rangle$ lies in the fact that for the latter calculation, the fundamental relations between

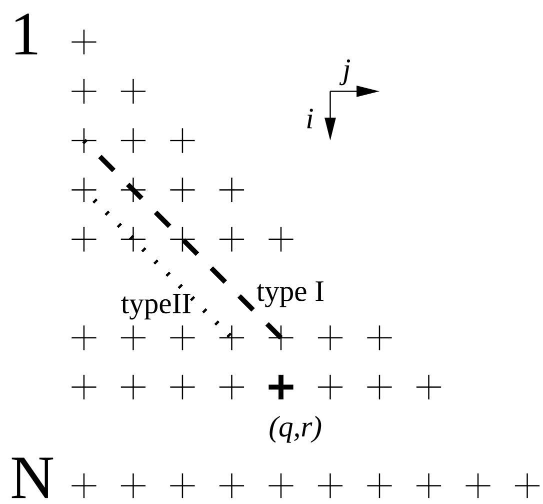

Figure 9. Propagation of defects in the recurrence relations for the calculation of $\langle W_{N-q,r}\rangle$.

the matrices are modified on the grains located at $(N-q-k, r-k)$, $0 \le k \le r-1$ and $(N-q-k, r-1-k)$, $0 \le k \le r-2$. The final result is of the form

$$\langle W_{N-q,r}\rangle = \frac{1}{\mathcal{N}} \left[\bar{L}^{(N-2)}\right]^{\mathrm{T}} \bar{t}^{2,1}(q,r)\, \bar{R}^{(N-2)}(q,r)\,, \tag{71}$$

but the explicit form of $\langle W_{N-q,r}\rangle$ depends on how the modified fundamental relations affect $\bar{L}^{(N-2)}$, $\bar{t}^{2,1}(q,r)$, and $\bar{R}^{(N-2)}(q,r)$, and hence on the values of r.

5.3.2. Calculation of $\langle W_{N-q,r}\rangle$ on the boundary, i.e., $r=1$: For $r=1$, there is only one relevant defect, and it is of type I. It affects only $\bar{L}^{(N-2)}$ and $\bar{t}^{2,1}(q,1)$, while $\bar{R}^{(N-2)}(q,1)$ remains the same as $R^{(N-2)}$. Let us first consider the case $q>1$, for which we have

$$\bar{t}^{2,1}(q,1) = \bar{\beta}^{(q,1)}_{i_1 j_1,\ldots,i_{N-2} j_{N-2}}\,. \tag{72}$$

Herein, the fundamental relation for $\bar{\beta}_{i_1 j_1,\ldots,i_{N-2} j_{N-2}}(q,1)$ gets modified only at the q-th layer to the form (66) to yield

$$\bar{\beta}^{q,1}_{i_1 j_1,\ldots,i_{N-2} j_{N-2}} = \beta_{i_1 j_1,\ldots,i_{N-2} j_{N-2}} \frac{q+1+i_{q-1}-j_{q-1}}{q+1+i_{q-1}-i_q}\,. \tag{73}$$

In addition, we should also keep in mind that the dimensions of the matrix $\bar{t}^{2,1}(q,1)$ are not the same as those of $t^{2,1}$, since the index i_q for $\bar{t}^{2,1}(q,1)$ varies between 1 and $q+1+i_{q-1}$ as opposed to varying between 1 and $q+i_{q-1}$ for $t^{2,1}$. This implies that the maximum value attained by i_l for $\bar{t}^{2,1}(q,1)$ for $l \ge q$ is increased by 1 due to the presence of the defect of type I at location $(N-q,r)$, and consequently

$$\bar{L}^{(N-2)}_{i_1,\ldots,i_{N-2}} = \prod_{l'=1}^{q-1} \delta_{i_{l'},\frac{l'(l'+1)}{2}+1} \prod_{l=q}^{N-2} \delta_{i_l,\frac{l(l+1)}{2}+2} \tag{74}$$

When Eqs. (72-74) are put together along with Eqs. (38) and (25) and (58) in Eq. (71), we obtain

$$\langle W_{N-q,1}\rangle = \frac{1}{\mathcal{N}}\sum_{j_1,\ldots,j_{N-2}} \frac{\left[\frac{q(q+1)}{2}+2-j_{q-1}\right]\Lambda^{(N-2)}_{j_1,\ldots,j_{N-2}}}{\left[\frac{N(N-1)}{2}+1\right]!}\,. \tag{75}$$

The two cases $q=0$ and $q=1$, however, have to be considered separately. For $q=1$, we find that Eq. (75) can be generalized by using $j_0=1$, yielding

$$\langle W_{N-1,1}\rangle = \frac{4}{(N-1)N+1} \tag{76}$$

while for $q=1$, it can be seen that $\langle W_{N,1}\rangle = \langle W_{N-1,1}\rangle/2$.

5.3.3. In the bulk, i.e., $r>1$: For $r>1$, each quantity in Eq. (71) is affected due to the development of the defects. Of these quantities, $\bar{t}^{2,1}(q,r)$ is affected by a defect of type I terminating on the boundary at location $(q+r-1,1)$ and a defect of type II at location $(q+r-2,1)$. Using Eqs. (66) and (69), we then get

$$\bar{\beta}^{q,r}_{i_1j_1,\ldots,i_{N-2}j_{N-2}} = \beta_{i_1j_1,\ldots,i_{N-2}j_{N-2}}\frac{q+r-1+j_{q+r-3}-j_{q+r-2}}{q+r+i_{q+r-2}-i_{q+r-1}}\,. \tag{77}$$

We also obtain, just as before,

$$\bar{L}^{(N-2)}_{i_1,\ldots,i_{(N-2)}} = \prod_{l'=1}^{q+r-2}\delta_{i_{l'},\frac{l'(l'+1)}{2}+1}\prod_{l=q+r-1}^{N-2}\delta_{i_l,\frac{l(l+1)}{2}+2}\,. \tag{78}$$

However, we still need to express $\bar{R}^{(N-2)}(q,r)$. Recall from Eq.(39) that $R^{(N-2)}$ depends, through the recursion formula, on $t^{N-p,N-p-1}$ for $1\le p\le N-1$. Since for the calculation of $\langle W_{N-q,r}\rangle$, $t^{N-p,N-p-1}$ for $N-r-1\le p\le N-3$ are modified due to the defects, $\bar{R}^{(N-2)}(q,r)$ must also differ from $R^{(N-2)}$.

More precisely, in the recursive formalism described above, $\bar{R}^{(p)}(q,r)$ does not differ from $R^{(p)}(q,r)$ for $p\le N-r-2$. The value $p=N-r-1$ corresponds to the layer number where a vertical line drawn from (q,r) in Fig. 9 intersects the right edge of the triangle.

Thus, up to $p=N-r-2$, the usual recursion formula applies so that

$$R^{(N-r-1)}_{i_1,\ldots,i_{N-r-1}} = \frac{\Lambda^{(N-r-1)}_{i_1,\ldots,i_{N-r-1}}}{(2-i_1)!(2+i_1-i_2)!\ldots(N-r-2+i_{N-r-3}-i_{N-r-2})!(N-r-2+i_{N-r-2})!}, \tag{79}$$

but then $\bar{\beta}^{r+1,r}$ is modified with respect to $\beta^{r+1,r}$ by a defect of type I at (q,r), and we get

$$\bar{R}^{(N-r-1)}_{i_1,\ldots,i_{N-r}} = \frac{\frac{1}{q+1+i_{q-1}-i_q}\bar{\Lambda}^{(N-r)}_{i_1\ldots i_{N-r}}(q,r)}{(2-i_1)!(2+i_1-i_2)!\ldots(N-r-1+i_{N-r-2}-i_{N-r-1})!(N-r-1+i_{N-r-1})!}, \tag{80}$$

where

$$\bar{\Lambda}^{(N-r)}_{i_1,\ldots,i_{N-r}}(q,r) = \sum_{j_1,\ldots,j_{N-r-1}}(q+1+i_{q-1}-j_{q-1})\prod_l\Theta(i_l-j_{l-1})\,\Lambda^{(N-r-1)}_{j_1\ldots j_{N-r-1}}. \tag{81}$$

Once again, it is important to remember that the q-th index i_q satisfies $1\le i_q\le q+1+i_{q-1}$, while all the others satisfy the inequalities (40).

At the next step of the recursion formula for $R^{(N-2)}$, $\bar{\beta}^{r,r-1}$ is modified with respect to $\beta^{r,r-1}$ by two defects: one of type I at $(q+1, r-1)$ and one of type II at $(q, r-1)$ and we obtain

$$\bar{R}^{(N-r+1)}_{i_1,\ldots,i_{N-r+1}} = \frac{\frac{1}{q+2+i_q-i_{q+1}}\bar{\Lambda}^{(N-r+1)}_{i_1\ldots i_{N-r+1}}(q,r)}{(2-i_1)!(2+i_1-i_2)!\ldots(N-r+i_{N-r-1}-i_{N-r})!(N-r+i_{N-r})!}, \tag{82}$$

where

$$\bar{\Lambda}^{(N-r+1)}_{i_1,\ldots,i_{N-r+1}}(q,r) = \sum_{j_1,\ldots,j_{N-r}} \prod_l \Theta(i_l - j_{l-1})\, \bar{\Lambda}^{(N-r)}_{j_1,\ldots,j_{N-r}}(q,r), \tag{83}$$

with $1 \le j_q \le q+1+j_{q-1}$ and $1 \le i_{q+1} \le q+2+i_q$. Notice that Eq. (83) is of the same form as Eq. (58) except for the bounds of j_q and i_{q+1}.

The calculations for $\bar{R}^{(N-k)}(q,r)$, $2 \le k \le r$ proceed along the same lines as for $\bar{R}^{(N-r+1)}(q,r)$ and finally we obtain

$$\bar{R}^{(N-2)}_{i_1,\ldots,i_{N-3}} = \frac{\frac{1}{q+r-1+i_{q+r-3}-i_{q+r-2}}\bar{\Lambda}^{(N-2)}_{i_1,\ldots,i_{N-2}}(q,r)}{(2-i_1)!(2-i_1-i_2)!\ldots(N-4+i_{N-5}-i_{N-4})!(N-4+i_{N-4})!}, \tag{84}$$

where $1 \le i_{q+r-2} \le q+r-1+i_{q+r-3}$, and for all the other indices, we have $1 \le i_l \le l+i_{l-1}$.

Putting everything together, as usual, most of the factorials that appear in the expressions of $\bar{t}^{2,1}(q,r)$ and $\bar{R}^{(N-2)}(q,r)$ cancel out and we are left with

$$\langle W_{N-q,r}\rangle = \frac{1}{\mathcal{N}} \sum_{j_1,\ldots,j_{N-2}} \frac{\bar{\Lambda}^{(N-2)}_{j_1,\ldots,j_{N-2}}(q,r)}{\left[\frac{N(N-1)}{2}+1\right]!}. \tag{85}$$

5.4. Higher moments and correlations

From the methods we described in Secs. 5.1-5.3, it is clear that higher moments and correlations can in principle be calculated by following the same procedure. We will not provide too many details below, instead we will demonstrate that the higher moments and correlations can easily be obtained by keeping track of more "defects" in the fundamental relations.

5.4.1. The case of $\langle W^s_{N-q,r}\rangle$, $s>1$ The modifications of the fundamental relations for the integration of $W^s_{N-q,r}$ are obtained by generalizing the calculations of section 5.3.1.

To start with, the $(i_1,\ldots,i_{q-1})$-th element of the matrix integration result $\int_{a_{N-q,r}}^{b_{N-q,r}} dW_{N-q,r} W^s_{N-q,r} t^{N-(q-1),k}\, \bar{J}^{N-q-1,r+1}(q,r)$ reads

$$\begin{aligned}
&\sum_{j_1,\ldots,j_q} \beta_{i_1 j_1,\ldots,i_{q-1}j_{q-1}} \gamma^{N-q,r+1}_{j_1,\ldots,j_q} I_{q+s+i_{q-1}-j_{q-1},q+j_{q-1}-j_q}(a_{N-q,r}, b_{N-q,r}) \\
&\qquad = \sum_{i_q=0}^{q+s+i_{q-1}} \bar{\gamma}^{N-q,r,s}_{i_1,\ldots i_q} (b_{N-q,r-1} - W_{N-q,r-1})^{q+s+i_{q-1}-i_q}
\end{aligned} \tag{86}$$

with $\bar{\gamma}^{N-q,r,s} = \bar{t}(q,r,s)^{N-q,r}\gamma^{N-q,r+1}$, where

$$\begin{aligned}
\bar{t}^{N-q,r}_{i_1 j_1,\ldots,i_q j_q}(q,r,s) &= \beta_{i_1 j_1,\ldots,i_{q-1}j_{q-1}} \Theta(i_q - j_{q-1})\, \alpha^{q+s+i_{q-1}-i_q}_{q+s+i_{q-1}-j_{q-1},q+j_{q-1}-j_q}\, W^{q+1+i_q-j_q}_{N-q-1,r-1} \\
&= \bar{\beta}^{(q,r,s)}_{i_1 j_1,\ldots,i_q j_q} W^{q+1+i_q-j_q}_{N-q-1,r-1}.
\end{aligned} \tag{87}$$

Once again, all the indices vary between the bounds defined in (40), except for i_q, which satisfies $1 \leq i_q \leq q+s+i_{q-1}$. The resulting modification of the fundamental relations can be called an s-th order defect of type I, and it is characterized by the relation

$$\bar{\beta}^{q,r,s}_{i_1 j_1,\ldots,i_q j_q} = \beta_{i_1 j_1,\ldots,i_q j_q} \frac{(q+s+i_{q-1}-j_{q-1})\ldots(q+1+i_{q-1}-j_{q-1})}{(q+s+i_{q-1}-i_q)\ldots(q+1+i_{q-1}-i_q)}. \quad (88)$$

After multiplying the r.h.s. of Eq.(86) with $t^{N-(q-1),r-1}$, the integral with respect to $W_{N-q,r-1}$ yields

$$\sum_{j_1,\ldots,j_q} \beta_{i_1 j_1,\ldots,i_{q-1} j_{q-1}}\, \bar{\gamma}^{N-q,r,s}_{j_1,\ldots,j_q}\, I_{q+i_{q-1}-j_{q-1},q+s+j_{q-1}-j_q}(a_{N-q,r-1}, b_{N-q,r-1})$$
$$= \sum_{i_q=0}^{q+i_{q-1}} \bar{\gamma}^{N-q,r-1,s}_{i_1\ldots i_q} (b_{N-q,r-2} - W_{N-q,r-2})^{q+i_{q-1}-i_q}. \quad (89)$$

Furthermore, using $\bar{\gamma}^{N-q,r-1,s} = \bar{t}^{N-q,r-1}(q,r,s)\,\bar{\gamma}^{N-q,r,s}$, we obtain

$$\bar{t}^{N-q,r-1,s}_{i_1 j_1,\ldots,i_q j_q}(q,r) = \beta_{i_1 j_1,\ldots,i_{q-1} j_{q-1}} \Theta(i_q - j_{q-1})\, \alpha^{q+i_{q-1}-i_q}_{q+i_{q-1}-j_{q-1},q+s+j_{q-1}-j_q} W^{q+s+1+i_q-j_q}_{N-q-1,r-1}, \quad (90)$$

where $1 \leq j_q \leq s+q+j_{q-1}$, while all the other indices satisfy Eq.(40). Thus, the s-th order defect of type II is characterized by

$$\bar{\beta}^{(q,r-1,s)}_{i_1 j_1,\ldots,i_q j_q} = \beta_{i_1 j_1,\ldots,i_q j_q} \frac{(q+s+j_{q-1}-j_q)\ldots(q+1+j_{q-1}-j_q)}{(q+s+1+i_q-j_q)\ldots(q+2+i_q-j_q)}. \quad (91)$$

As in the case for $s=1$, all the other integrals on the $(N-q)$-th layer have the usual form (43), so that there are only two defects per layer to take care of. Moreover, these defects propagate exactly in the same as shown in Fig. 9(a).

The explicit expression for $\langle W^s_{N-q,r} \rangle$ in terms of $\bar{\Lambda}$'s can be directly deduced from the recurrence relations for the $\bar{t}$'s.

5.4.2. Correlations of the type $\langle W^{s_1}_{N-q_1,r_1} \ldots W^{s_m}_{N-q_m,r_m} \rangle$: It turns out that these quantities can be calculated using the results for $\langle W^s_{N-q,r} \rangle$.

Consider the simplest case $\langle W^{s_1}_{N-q_1,r_1} W^{s_2}_{N-q_2,r_2} \rangle$ with $q_2 \geq q_1$. If the point (q_2, r_2) does not lie on one of the two lines of defects originating from (q_1, r_1), it is clear that the effects on the recurrence relations of the defects originating from (q_1, r_1) and (q_2, r_2) do not "interact" with each other.

The defects do "interact" only if (q_2, r_2) lies on the line defined by (q_1+k, r_1-k) for $0 \leq k \leq r_1-1$. In that case, the defects are of s_1-th order of type I on (q_1+k, r_1-k), $0 \leq k \leq r_1-r_2-1$, and of type II on (q_1+k, r_1-k-1), $0 \leq k \leq r_1-r_2-2$; and then of (s_1+s_2)-th order of type I on (q_1+k, r_1-k),$r_1-r_2 \leq k \leq r_1-1$. and type II (q_1+k, r_1-k-1), $r_1-r_2 \leq k \leq r_1-2$.

The modifications of the fundamental relations for all the higher correlations $\langle W^{s_1}_{N-q_1,r_1} \ldots W^{s_m}_{N-q_m,r_m} \rangle$ can be obtained by using these observations. Once again if the lines of defects originating from the points (q_i, r_i) for $i = 1, \ldots, m$ do not intersect, the individual defects do not "interact". If they do intersect, the defects "interact" as described above, implying that all correlations can be expressed in terms of $\bar{\Lambda}$'s.

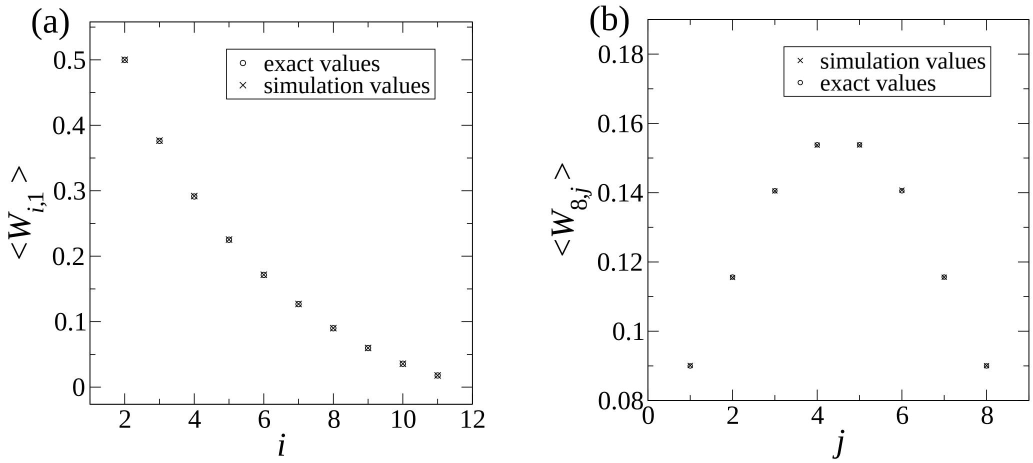

Figure 10. Comparison between exact results and simulations: (a) on the boundary $r = 1$ for $N = 11$; (b) in the bulk for $q = 3$ and $N = 11$.

5.5. Comparison with simulation results

To evaluate $\langle W_{i,j}\rangle$'s exactly using the exact relations developed above for any system size, we need to compute the integers $\Lambda^{(p)}_{i_1\ldots i_p}$ and correspondingly the $\bar{\Lambda}^{(p)}_{i_1\ldots i_p}$. These integers are defined recurrently by the relations (58). Although the relations (58) are simple sums, the number of terms necessary to evaluate $\Lambda^{(p)}_{i_1\ldots i_p}$ increases as roughly as $(p!)!$ for large p, so that for practical purposes, it is difficult to go beyond $N = 11$. The comparison between the exact evaluation of the $\langle W_{i,j}\rangle$'s with the corresponding Monte-Carlo simulation results for $N = 11$ is shown in Fig. 10.

5.6. An equivalent combinatorial problem

We have not been able to find an explicit expression or an asymptotic formula for $\Lambda^{(p)}_{i_1,\ldots,i_p}$ for large p. However, after some relabeling, a combinatorial interpretation can be obtained for $\Lambda^{(p)}_{i_1,\ldots,i_p}$ and related expressions entering the formulas for the moments of $W_{i,j}$.

This interpretation goes as follows: we consider maps h which associate a positive integer $h_{(k,l)}$ to the (k,l)-th site of the portion of the square lattice defined by $1 \le k \le p$ and $1 \le l \le k$ [i.e., the triangle in Fig. 8(a)]. The integers $h_{(k,l)}$ are constrained by the following inequalities:

$$\begin{aligned} & h_{(1,1)} = 1, \\ & 1 \le h_{(l,1)} \le 2, \quad \forall l = 2\ldots p \\ & 1 \le h_{(l,k)} \le k + h_{(l,k-1)}, \quad \forall l = 2\ldots p, k = 1\ldots l. \end{aligned} \tag{92}$$

The inequalities (92) are in fact just a re-expression of Eq. (40). Furthermore, Eq. (58) can then be rewritten as

$$\Lambda^{(2)}_{h_{(11)}h_{(12)}} = \delta_{h_{(12)},1}$$

$$\Lambda^{(p)}_{h_{(p,1)}\dots h_{(p,p)}} = \delta_{h_{(p,p)},1} \sum_{h_{(p-1,1)},\dots,h_{(p-1,p-1)}} \Lambda^{(p-1)}_{h_{(p-1,1)},\dots,h_{(p-1)}} \prod_{l=1}^{p-1} \Theta[h_{(p,l+1)} - h_{(p-1,l)}]. \quad (93)$$

Having iterated the recurrence relation, an equivalent "explicit" expression for Λ's can be obtained as

$$\Lambda^{(p)}_{h_{(p,1)}\dots h_{(p,p)}} = \sum_{\{h\}} \prod_{k=1}^{p} \prod_{l=1}^{k} \Theta[h_{(k,l+1)} - h_{(k-1,l)}]\,, \quad (94)$$

where the sum runs over all maps h satisfying Eq. (92) with fixed values of $h_{(p,k)}$ for $k = 1, \dots, p$. The products on the right hand side are obviously non-zero only for maps h satisfying

$$h_{(k,l)} \leq h_{(k+1,l+1)} \quad \forall k = 1 \dots p-1, l = 1, \dots, k. \quad (95)$$

We now introduce a new symbol Z_{N-2} for $\sum_{j_1,\dots,j_{N-2}} \Lambda^{(N-2)}_{j_1,\dots,j_{N-2}}$, which enters the expression of the normalization constant $\mathcal{N}$. It is then easily seen that Z_{N-2} is simply the total number of maps h satisfying inequalities (92) and (95) for $p = N-2$. In a similar manner, the quantity $\sum_{j_1,\dots,j_{N-2}} \bar{\Lambda}^{(N-2)}_{j_1,\dots,j_{N-2}}(q,r)$ from (85) can be re-expressed in terms of maps $\bar{h}(q,r)$ as $\sum_{\{\bar{h}(q,r)\}}[q+1+\bar{h}_{(N-r,q-1)} - \bar{h}_{(N-r-1,q-1)}]$ where the maps $\bar{h}(q,r)$ satisfy the inequalities (92) and (95) except on the line $(l,k) = (N-r+j, q+j)$ for $0 \leq j \leq r-2$, where they satisfy

$$1 \leq \bar{h}_{(l,k)} \leq k + 1 + \bar{h}_{(l,k-1)}. \quad (96)$$

Thus, we finally get

$$\langle W_{N-q,r} \rangle = \frac{1}{\frac{N(N-1)}{2} + 1} \frac{\bar{Z}_{N-2}(q,r)}{Z_{N-2}} \langle q + 1 + \bar{h}_{(N-r,q-1)} - \bar{h}_{(N-r-1,q-1)} \rangle_{\bar{h}(q,r)}, \quad (97)$$

where $\bar{Z}_{N-2}(q,r)$ is the total number of maps $\bar{h}(q,r)$ for $p = N-2$, and the angular brackets on the r.h.s. denote an average over all maps. Similar expressions can also be obtained for all higher moments and correlations.

Interestingly, the original symmetry of the model is not apparent at all in this formulation. Although the final results, once calculated [cf. Fig.10], display of course the symmetry, it seems difficult to show that the underlying discrete problem is indeed symmetric.

6. Discussion and Conclusions

In this paper, we have studied the response of a hexagonal packing of rigid, frictionless, massless, spherical grains to a single external force at the top of it, by supposing all mechanically stable force configurations equally likely. We have shown that this problem is equivalent to a correlated q-model. Interestingly, while the conventional q-model produces a single-peaked, diffusive response, our model leads to two sharp peaks on the boundary, i.e., on the two lattice directions emanating from the point of application of the external force. For systems of finite size, the magnitude of these peaks decreases towards the bottom of the packing, while progressively a broader, central

maximum appears between the peaks. The response function displays a remarkable scaling behaviour with system size N: while the response in the bulk of the packing scales as $\frac{1}{N}$, on the boundary it is independent of N, so that in the thermodynamic limit only the peaks on the lattice directions persist. We have obtained exact expressions of the $\langle W_{i,j} \rangle$ values, i.e., of the response, and of higher correlations for any system size in terms of integers corresponding to an underlying discrete structure, but we have not been able to derive an expression for the scaling limit. The resemblance of the discrete structure with plane-partition problems [29] might lead to further progress in that direction.

Qualitatively, the response obtained via the uniform probability hypothesis is thus in agreement with experiments. It should however be noted that in experiments, the width of the peaks increases linearly with depth, while in our model, the peaks are single-grain diameter wide at any depth. Such peak widening, in our opinion, is a consequence of inter-grain friction. Indeed, a recent study [28] has shown that friction in rectangular packings produces a widening of the peaks around the two lattice directions emanating from the point of application of the single external force, with the peak width proportional to the square root of depth. However, in Ref. [28], the various force configurations have not been sampled uniformly as in our model, but rather the sampling was carried out in a fashion similar to the usual q-model, with independent random values at each grain. We are now attempting to study the effect of friction within the uniform ensemble in both rectangular and hexagonal geometries; but our preliminary attempts reveal that such a study is technically difficult as the mapping to an analogous q-formulation breaks down.

A natural question that arises in view of our work is whether the uniform probability hypothesis leads to one of three categories that the existing continuum-type models for the transmissions of stresses have been classified into (namely, elliptic, hyperbolic and parabolic according to the nature of the underlying coarse-grained PDE's) [2]. Given the critical importance of the underlying network of contacts, it seems difficult to give a general answer to this question. For the hexagonal geometry that we studied, it is possible to write a coarse-grained equation for vertical stresses ω using the q-formulation; indeed, since the correlation length between the q's is rather short, the continuous limit is the same is in the usual q-model [13]

$$\partial_z \omega + \partial_x (v\omega) = D_0 \partial_{xx} \omega \,, \tag{98}$$

where $v(x,z)$ is the noise resulting from the continuous limit of $q_{i,j} = (1 + v_{i,j})/2$. However, in contrast with the usual q-model, the mean $\langle v(x,z) \rangle$ is position-dependent as found in Sec. 3.3. Such a formulation thus does not seem very useful, as the properties of the noise and the self-similarity have to be somehow first obtained from the equal-probability ensemble hypothesis. Nevertheless, the study of a tensorial formulation of this model is an interesting future direction.

Finally, apart from the inclusion of friction in the present model, the next important future direction is the study of the response of disordered granular packings. In that

case, averages would have to be taken first for a fixed contact network, and then over different contact networks corresponding to different arrangement geometries of grains. The shape of the resulting response function would provide another crucial test for the applications of the equal probability hypothesis to the study of forces in granular packings.

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